# Bicategory of Theories as an Approach to Model Theory

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# Introduction

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- definability theorem and duality theory (Makkai 1993)

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In terms of FOCL, some classical model-theoretic phenomena can be rephrased and generalized:

- Gödel's completeness theorem vs. Deligne's theorem
- definability theorem and duality theory (Makkai 1993)

However, most concepts in *modern* model theory have not been considered categorically, e.g.,

stability, non-forking extensions and saturation.

## Introduction: Backgrounds of FOCL

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### Classical Observation (*cf.* Makkai 1993)

The $2$-category $\mathfrak{BPretop}_*$ of small Boolean pretoposes, pretopos functors and natural isomorphisms can be seen as a “$2$-category of classical first-order theories.”

This viewpoint is discussed in [Halvorson and Tsementzis 2016].

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The $2$-category $\mathfrak{BPretop}_*$ of small Boolean pretoposes, pretopos functors and natural isomorphisms can be seen as a "$2$-category of classical first-order theories."

This viewpoint is discussed in [Halvorson and Tsementzis 2016]. We will give a conclusive evidence for this viewpoint. Our result is built upon many previous works, which include [Pitts 1989; Visser 2006; Tsementzis 2015].

# Introduction: Overview

## Our Contributions

- Proposing a new bicategory $\mathfrak{Th}$ of theories, interpretations and homotopies, which is built up purely syntactically.
- Improving previous works and integrating them as a biequivalence between $\mathfrak{Th}$ and $\mathfrak{BPretop}_*$.
- Characterizing bi-interpretability both syntactically and categorically (as well as improving the work of [Tsementzis 2015]).

| $\mathfrak{Th}$ | $\mathfrak{BPretop}_*$ |
|---|---|
| theory | Boolean pretopos |
| interpretation | pretopos functor |
| homotopy | natural isomorphism |
| bi-interpretability | Morita equivalence |

# Contents

## Contents of the Current Section

## Syntactic Categories I

We consider **many-sorted classical first-order theories**.
Let $T$ be an $\mathcal{L}$-theory.
Below is a generalization of Lindenbaum-Tarski algebra:

**Definition**

The **syntactic category** $\mathcal{C}_T$ of $T$ consists of:

Objects: $\mathcal{L}$-formulas-in-context $\{\boldsymbol{x}.\,\varphi\}$

Morphisms: $T$-provably functional formulas $[\chi]\colon \{\boldsymbol{x}.\,\varphi\} \to \{\boldsymbol{y}.\,\psi\}$

Then, $\mathcal{C}_T$ is a **Boolean coherent category**, i.e.,

- All finite limits exist.
- Each morphism has an image factorization.
- Any subobject poset $\mathrm{Sub}(\{\boldsymbol{x}.\,\varphi\})$ is a Boolean algebra.
- These categorical structures are stable under pullbacks.

## Syntactic Categories II

### Proposition

Each $T$-model $\mathcal{M}$ gives a **coherent functor** $F_{\mathcal{M}} \colon \mathcal{C}_T \to \mathbf{Set}$ which sends $\{\boldsymbol{x}.\,\varphi\}$ to the definable set $\varphi(\mathcal{M}) \subseteq |\mathcal{M}|^{\boldsymbol{x}}$.
Moreover, this correspondence yields (part of) an equivalence:

$$\mathbf{Elem}(T) \simeq \mathfrak{Coh}(\mathcal{C}_T, \mathbf{Set})$$

where

- $\mathbf{Elem}(T)$ is the category of $T$-models and elementary embeddings, and
- $\mathfrak{Coh}(\mathcal{C}_T, \mathbf{Set})$ is the category of coherent functors from $\mathcal{C}_T$ to $\mathbf{Set}$ and natural transformations.

# Shelah's $\mathrm{eq}$-Construction and Classifying Pretoposes I

## Shelah's $\mathrm{eq}$-Construction

Put $\mathcal{L}^{\mathrm{eq}} = \mathcal{L} \cup \{\, S_\Delta \,;\, \Delta$ is a $T$-equivalence relation on a type $\bar{A}\,\}$
$\cup \{\, \varepsilon_\Delta \colon \bar{A} \to S_\Delta \,;$ for each $\Delta\,\}$,
and $T^{\mathrm{eq}} = T \cup \{$ "$S_\Delta$ is the quotient $\bar{A}/\Delta$" $\}$.
Any $T$-model $\mathcal{M}$ is canonically expanded to a $T^{\mathrm{eq}}$-model $\mathcal{M}^{\mathrm{eq}}$.

We now consider the category $\mathcal{C}_{T^{\mathrm{eq}}}$.

## Theorem (Harnik 2011, §5)

Under a mild condition on $T$, $\mathcal{C}_{T^{\mathrm{eq}}}$ becomes a (Boolean) **pretopos**, i.e., a (Boolean) coherent category in which:

- Any "equivalence relation" has a quotient. (exact cat.)
- Finite coproducts exist and are disjoint. (extensive cat.)

# Shelah's $\mathrm{eq}$-Construction and Classifying Pretoposes II

## Theorem (cont'd)

The canonical functor $\iota\colon \mathcal{C}_T \to \mathcal{C}_{T^{\mathrm{eq}}}$ is a **pretopos completion**:

- for any pretopos $\mathcal{P}$, and
- for any coherent functor $F\colon \mathcal{C}_T \to \mathcal{P}$,

there exists a unique pretopos (=coherent) functor $G\colon \mathcal{C}_{T^{\mathrm{eq}}} \to \mathcal{P}$ (up to natural isomorphism) such that

$$\begin{tikzcd} \mathcal{C}_T \arrow[r, "\iota"] \arrow[dr, "F"'] & \mathcal{C}_{T^{\mathrm{eq}}} \arrow[d, dashed, "G"] \\ & \mathcal{P} \end{tikzcd}$$

$$\mathfrak{Coh}(\mathcal{C}_T, \mathcal{P}) \simeq \mathfrak{Coh}(\mathcal{C}_{T^{\mathrm{eq}}}, \mathcal{P}),$$

(hence, $\mathbf{Elem}(T) \simeq \mathbf{Elem}(T^{\mathrm{eq}})$).

$\mathcal{P}_T := \mathcal{C}_{T^{\mathrm{eq}}}$: the **classifying pretopos** of $T$

| $\mathfrak{Th}$ | $\mathfrak{BPretop}_*$ |
|---|---|
| Shelah's $\mathrm{eq}$-construction | pretopos completion |
| theory | Boolean pretopos |

$$\mathfrak{Th} \qquad\qquad\qquad \mathfrak{BPretop}_*$$

$$\boxed{\quad T \qquad\qquad T' \quad} \longrightarrow \boxed{\quad \mathcal{P}_T \qquad\qquad \mathcal{P}_{T'} \quad}$$

## Interpretations I

Let $T$ (resp. $T'$) be an $\mathcal{L}$-theory (resp. an $\mathcal{L}'$-theory).

### Definition (*cf.* Hodges 1993, Ch. 5 §3)

A **pre-interpretation** $I$ of $T$ in $T'$ sends

$$\mathcal{L}\text{-sort } A \quad \rightsquigarrow \quad \text{a pair } (\partial^I_A, \Delta^I_A) \text{ with } \begin{cases} \partial^I_A : \mathcal{L}'\text{-formula} \\ \Delta^I_A : T'\text{-equivalence relation on } \partial^I_A \end{cases}$$

$$\mathcal{L}\text{-relation } R \rightarrowtail \bar{A} \quad \rightsquigarrow \quad \mathcal{L}'\text{-formula } R^I \subseteq \partial^I_{\bar{A}} \text{ which is closed under } \Delta^I_{\bar{A}}.$$

$$\mathcal{L}\text{-function } f \colon \bar{A} \to B \quad \rightsquigarrow \quad \mathcal{L}'\text{-formula } \Gamma^I_f \subseteq \partial^I_{\bar{A}B} \text{ which induces “a morphism } \partial^I_{\bar{A}}/\Delta^I_{\bar{A}} \to \partial^I_B/\Delta^I_B\text{.”}$$

## Interpretations II

$I$ induces a map $\varphi \mapsto \varphi^I \subseteq \partial^I_{\bar{A}}$.

**Definition**

A pre-interpretation $I$ of $T$ in $T'$ is said to be an **interpretation** (denoted by $I\colon T \to T'$) if, for any $\mathcal{L}$-sentence $\varphi$,

$$\varphi \in T \quad \text{implies} \quad T' \models \varphi^I.$$

Each $T'$-model induces a $T$-model (Use appropriate quotients).

**Example**

$\mathrm{ACF}_0$ is interpretable in $\mathrm{RCF}$ by

$$\begin{array}{ccc} s = s & \rightsquigarrow & x = x \wedge y = y \\ s + t & \rightsquigarrow & (x_1 + x_2, y_1 + y_2) \\ s \cdot t & \rightsquigarrow & (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2) \end{array}$$

Hence, for any real closed field $R$, we can obtain a field of "complex numbers" by defining operations on $R^2$.

# Contents of the Current Section

## Interpretation as a Pretopos Functor

- A formal quotient "$\varphi^I/\Delta^I_{\bar{A}}$" turns into a genuine $\mathcal{L}'^{\mathrm{eq}}$-formula. Hence, an interpretation can be seen as a map $\varphi \mapsto \varphi^I/\Delta^I_{\bar{A}}$.
- This map can be extended to a coherent functor $F_I \colon \mathcal{C}_T \to \mathcal{C}_{T'^{\mathrm{eq}}}$.
- Finally, we obtain a pretopos functor $\mathcal{P}_I \colon \mathcal{P}_T \to \mathcal{P}_{T'}$ by the universality of pretopos completion $\mathcal{P}_T$.

$$
\begin{tikzcd}
\mathcal{C}_T \arrow[r, "\iota"] \arrow[dr, "F_I"'] & \mathcal{C}_{T^{\mathrm{eq}}} = \mathcal{P}_T \arrow[d, dashed, "\mathcal{P}_I"] \\
& \mathcal{C}_{T'^{\mathrm{eq}}} = \mathcal{P}_{T'}
\end{tikzcd}
$$

| $\mathfrak{Th}$ | $\mathfrak{BPretop}_*$ |
|---|---|
| Shelah's $\mathrm{eq}$-construction | pretopos completion |
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$\mathfrak{Th}$ $\qquad\qquad\qquad\qquad$ $\mathfrak{BPretop}_*$

$$\begin{tikzcd} T \arrow[r, bend left, "I"] \arrow[r, bend right, "J"'] & T' \end{tikzcd} \quad\longrightarrow\quad \begin{tikzcd} \mathcal{P}_T \arrow[r, bend left, "\mathcal{P}_I"] \arrow[r, bend right, "\mathcal{P}_J"'] & \mathcal{P}_{T'} \end{tikzcd}$$

## Bicategory $\mathfrak{Th}$

Let $I\colon T \to T'$, $J\colon T' \to T''$ be interpretations.
Then the composite $JI\colon T \to T''$ can be defined canonically.

Let $I, J, K$ be a composable triple of interpretations. For syntactic reasons, two composites $K(JI)$ and $(KJ)I$ do not necessarily coincide on the nose.

These interpretations are **homotopic** in an appropriate sense (*cf.* Hodges 1993, Ch. 5 §4).

### Definition

$\mathfrak{Th}$ denotes the *bicategory* of theories, interpretations and homotopies.

## Construction of the Biequivalence I

A homotopy between interpretations induces a natural isomorphism between corresponding pretopos functors, and hence we have a pseudofunctor $\mathfrak{Th} \to \mathfrak{BPretop}_*$.

$$\mathfrak{Th} \qquad\qquad \mathfrak{BPretop}_*$$

$$\boxed{\begin{tikzcd} T \arrow[r, bend left, "I"] \arrow[r, bend right, "J"'] \arrow[r, phantom, "\Downarrow h"] & T' \end{tikzcd}} \longrightarrow \boxed{\begin{tikzcd} \mathcal{P}_T \arrow[r, bend left, "\mathcal{P}_I"] \arrow[r, bend right, "\mathcal{P}_J"'] \arrow[r, phantom, "\Downarrow \mathcal{P}_h"] & \mathcal{P}_{T'} \end{tikzcd}}$$

## Construction of the Biequivalence II

- Pay attention to the functor

$$\mathfrak{Th}(T, T') \to \mathfrak{BPretop}_*(\mathcal{P}_T, \mathcal{P}_{T'}) \qquad I \mapsto \mathcal{P}_I.$$

  This gives an equivalence between these hom-categories.
- Any small Boolean pretopos $\mathcal{P}$ is categorically equivalent to the classifying pretopos $\mathcal{P}_T$ of some theory $T$.

## Construction of the Biequivalence II

- Pay attention to the functor
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**Theorem (A.)**

$\mathfrak{Th}$ and $\mathfrak{BPretop}_*$ are biequivalent.

| $\mathfrak{Th}$ | $\mathfrak{BPretop}_*$ |
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| theory | Boolean pretopos |
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| homotopy | natural isomorphism |

## Contents of the Current Section

# Bi-interpretability

## Definition

We say that $T$ and $T'$ are **bi-interpretable** when there exist two interpretations $I\colon T \to T'$, $J\colon T' \to T$ such that

- $JI$ is homotopic to $\mathrm{id}_T\colon T \to T$ (the identity interpretation),
- $IJ$ is homotopic to $\mathrm{id}_{T'}\colon T' \to T'$.

Completeness, stability and $\kappa$-categoricity are preserved under bi-interpretability.

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Completeness, stability and $\kappa$-categoricity are preserved under bi-interpretability.

## Theorem (A.)

$T$ and $T'$ are bi-interpretable precisely when their classifying pretoposes are equivalent, i.e., $\mathcal{P}_T \simeq \mathcal{P}_{T'}$ (Morita equivalence).

This immediately follows from the biequivalence. Can we construct a bi-interpretation more concretely from Morita equivalence?

# Morita Extension

## Definition (Barrett and Halvorson 2016)

(1) An extension of theories $T \subseteq T'$ (not necessarily in the same language) is a **Morita extension** when $T'$ is obtained by adding to $T$ some explicit definitions and **sort definitions**.

(2) A **Morita span** from $T$ to $T'$ consists of a sequence of Morita extensions of the following form:

$$
\begin{tikzcd}
& & & \text{logically equivalent} & & & \\
& & & T_n = T'_m & & & \\
& & \cdots \arrow[ur, phantom, "\subseteq" sloped] & & \cdots \arrow[ul, phantom, "\supseteq" sloped] & & \\
& T_1 \arrow[ur, phantom, "\subseteq" sloped] & & & & T'_1 \arrow[ul, phantom, "\supseteq" sloped] & \\
T \arrow[r, leftrightarrow, "\text{copy}"] & T_0 \arrow[u, phantom, "\cup|"] & & & & T'_0 \arrow[u, phantom, "\cup|"] \arrow[r, leftrightarrow, "\text{copy}"] & T'
\end{tikzcd}
$$

# Previous Work

**Theorem (Tsementzis 2015)**

The following are equivalent:

(i) $\mathcal{P}_T \simeq \mathcal{P}_{T'}$ (Morita equivalence).

(ii) There exists a Morita span from $T$ to $T'$.

He actually constructed a sequence of Morita spans from Morita equivalence, but he did not make clear how to get a single Morita span.

We give a clear and rigorous construction of a Morita span from Morita equivalence.

## Characterization Theorem

### Theorem

The following are equivalent:

(i) $T, T'$ are bi-interpretable.

(ii) $\mathcal{P}_T \simeq \mathcal{P}_{T'}$ (Morita equivalence).

(iii) There exists a Morita span from $T$ to $T'$.

(i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i): Immediate. To see (ii) $\Rightarrow$ (iii), we give an explicit construction of a Morita span from Morita equivalence.

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(i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i): Immediate. To see (ii) $\Rightarrow$ (iii), we give an explicit construction of a Morita span from Morita equivalence.

- Many model-theoretic properties (e.g. completeness, stability and $\kappa$-categoricity) are Morita-invariant.
- For a model-theoretic property $P$, showing Morita-invariance of $P$ reduces to invariance under Morita extensions.

# Contents of the Current Section

## Model-theoretic Properties of a Pretopos

### Definition

A Boolean pretopos is **stable** if it is equivalent to $\mathcal{P}_T$ for some stable theory $T$.

### Questions

- Can we describe stability of a pretopos purely categorically? (*cf.* completeness of $T$ vs. two-valuedness of $\mathcal{P}_T$.)
- Is stability closed under categorical constructions for pretoposes?

These questions can considered for any other *model-theoretic properties* of a Boolean pretopos (possibly, of a general pretopos).

# Model-theoretic Constructions, $2$-Categorically

Some constructions of theories can be described $2$-categorically:

| $\mathfrak{Th}$ | $\mathfrak{BPretop}_*$ |
|---|---|
| add axioms in the same language | **quotient** of a pretopos |
| add a new constant | **slice category** of a pretopos |

## Questions

- What about elementary diagrams of models, special extensions of types and other constructions?
- Can we use category theory to find new constructions for theories?

Combining these directions, we will explore more comprehensive categorical analysis of model theory. These techniques might be used in non-classical situations (including the infinitary case).

# Summary

| $\mathfrak{Th}$ | $\mathfrak{BPretop}_*$ |
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| Shelah's $\mathrm{eq}$-construction | pretopos completion |
| theory | Boolean pretopos |
| interpretation | pretopos functor |
| homotopy | natural isomorphism |
| bi-interpretability | Morita equivalence |

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