$\mathcal{Q}$-sup-algebras and their representation

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Outline

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Introduction - Q-valued structures

Stubbe constructed an isomorphism between the categories of right Q-modules and cocomplete skeletal Q-categories for a given unital quantale Q. Employing his results, Solovyov obtained an isomorphism between the categories of Q-algebras and Q-quantales, where Q is additionally assumed to be commutative (a lecture on a non-commutative version of this result was presented at 92nd Workshop on General Algebra in 2016).

Resende introduced (many-sorted) sup-algebras that are certain partially ordered algebraic structures which generalize quantales, frames and biframes (pointless topologies) as well as various lattices of multiplicative ideals from ring theory and functional analysis (C*-algebras, von Neumann algebras). One-sorted case was studied e.g. by Zhang and Laan, J.P., and, in the generalized form of Q-sup-algebras by Šlesinger.

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Introduction - Q-valued structures

With this paper we hope to contribute to the theory of quantales and quantale-like structures. It considers the notion of Q-sup-algebra and shows a representation theorem for such structures generalizing the well-known representation theorems for quantales and sup-algebras. In addition, we present some important properties of the category of Q-sup-algebras.

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Sup-lattices

- ***Sup-lattice***: a partially ordered set (complete lattice) in which every subset S has a join (supremum) $\bigvee S$, and therefore also a meet (infimum) $\bigwedge S$. The greatest element is denoted by $\top$, the least element by $\bot$.
- ***Sup-lattice homomorphism***: join-preserving mapping, $f(\bigvee S) = \bigvee\{f(s) \mid s \in S\}$ (preserves $\bot$)

  $\Longrightarrow$ category $\mathbf{Sup}$.
- A mapping $f : A \to B$ if a sup-lattice homomorphism if and only if it has a ***right adjoint*** $g : B \to A$, by which is meant a mapping g that satisfies

$$f(a) \leq b \iff a \leq g(b)$$

for all $a \in A$ and $b \in B$. We write $f \dashv g$ in order to state that g is a right adjoint to f (equivalently, f is a left adjoint to g).

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Quantales

- ***Quantale***: is a semigroup object in the monoidal category of sup-lattices in $\mathbf{Sup}$ (associative *multiplication* $\cdot$ distributes over joins in both arguments)

$$a \cdot (\bigvee S) = \bigvee_{b \in S} a \cdot b\,, \quad (\bigvee S) \cdot a = \bigvee_{b \in S} b \cdot a$$

  for all $a \in Q$ and $S \subset Q$.
- ***Unital quantale***: has a multiplication unit (need not be the greatest element), denoted 1.
- For any $a \in Q$, $a \cdot -$ preserves joins $\implies$ has a right adjoint $a \to_r -$, satisfying $a \cdot b \leq c \iff b \leq a \to_r c$ for any $a, b, c \in Q$, explicitly $a \to_r c = \bigvee \{b \in Q \mid a \cdot b \leq c\}$
- By analogy we have $a \to_l -$ for $- \cdot a$ (for Q commutative the operations coincide, denoted $\to$).

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Quantale homomorphisms

- ***Quantale homomorphism***: is a homomorphism $f : Q \to R$ of semigroups that is also a homomorphism of sup-lattices; that is, for all $S \subseteq Q$ and $a, b \in Q$ we have

$$f(\bigvee S) = \bigvee_{s \in S} f(s)\,, \quad f(a \cdot b) = f(a) \cdot f(b)\,.$$

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- A homomorphism of unital quantales is ***unital*** if it is a monoid homomorphism.

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Quantale modules

Given a quantale Q.

- (Left) ***module over*** Q: a sup-lattice M with a (left) action $- * -: Q \times M \to M$ of Q, which distributes over joins in both arguments (right modules defined by analogy)
- Adjoints $\to_Q : M \times M \to Q$ and $\to_M : Q \times M \to M$
- Unital module: for Q unital, such that $1 * m = m$ for all $m \in M$.
- Q-***module homomorphism*** from M to N: is a $\bigvee$-preserving map $\varphi : M \to N$ with $\varphi(q * a) = q * \varphi(a)$ for every $q \in Q, a \in M$.

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1. Introduction - Q-valued structures
2. Basic notions, definitions and results
   - Sup-lattices
   - Quantales
   - Quantale modules
3. Q-sup-lattices
   - Q-order
   - Fuzzy sets
   - Q-sup-lattices
   - Q-**Sup** is isomorphic to Q-**Mod**
4. Q-sup-algebras
   - Po-algebras and sup-algebras
   - Q-po-algebras and Q-sup-algebras
   - Representation theorem for Q-sup-algebras

Q-order

Let Q be a fixed unital (commutative) quantale.

Definition

Let X be a set. A mapping $e\colon X \times X \to Q$ is a ***Q-order*** if

1. $e(x,x) \geq 1$, (Q-reflexivity)
2. $e(x,y) \cdot e(y,z) \leq e(x,z)$, ($Q$-transitivity)
3. $e(x,y) \geq 1$ and $e(y,x) \geq 1 \implies x = y$ (Q-antisymmetry)

for any $x, y, z \in X$.

Instance of fuzzy relation (introduced by Zadeh 1971).

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At the same time Fan and Zhang, in order to study quantitative domains theory under the framework of fuzzy set theory, for L a complete Heyting algebra, defined and studied a kind of L-orders, called the degree functions.

In fact, an L-order in the sense of Fan-Zhang (when being extended onto a complete residuated lattice) and an L-order in the sense of Bělohlávek are equivalent to each other. Both orders are special kinds of Ω-categories, and a more general notion is that of categories enriched in a quantaloid.

Example

1. $Q = 2$: partial order relation
2. A (left) Q-module M: $e(m,n) = m \to_Q n = \bigvee\{q \in Q \mid qm \leq n\}$

Induced partial order relation: $x \leq_e y$ iff $e(x,y) \geq 1$.

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2. A (left) Q-module M: $e(m,n) = m \to_Q n = \bigvee\{q \in Q \mid q * m \leq n\}$

Induced partial order relation: $x \leq_e y$ iff $e(x,y) \geq 1$.

Q-order

At the same time Fan and Zhang, in order to study quantitative domains theory under the framework of fuzzy set theory, for L a complete Heyting algebra, defined and studied a kind of L-orders, called the degree functions.

In fact, an L-order in the sense of Fan-Zhang (when being extended onto a complete residuated lattice) and an L-order in the sense of Bělohlávek are equivalent to each other. Both orders are special kinds of Ω-categories, and a more general notion is that of categories enriched in a quantaloid.

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Q-monotone mappings

Definition

Let (X, e_X) and (Y, e_Y) be Q-ordered sets. A mapping $f\colon X \to Y$ is ***Q-monotone*** if

$$e_X(x,y) \le e_Y(f(x), f(y))$$

for any $x, y \in X$.

$\implies$ category Q-**Ord**.

Every Q-monotone mapping is monotone wrt. to the induced partial order $\le_e$.

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Q-suprema

Definition

A ***Q-subset*** of a set X is an element of the set Q^X.

Again, for $Q = 2$ coincides with ordinary subsets.

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   or, equivalently,
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For $A \subseteq X$ and $M(x) = \begin{cases} 1 & \text{if } x \in A, \\ \perp & \text{otherwise,} \end{cases}$ we have $\bigsqcup M = \bigvee A$ whenever $\bigsqcup M$ or $\bigvee A$ is defined.

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Q-sup-lattices

Definition

A Q-ordered set (X, e) is a ***Q-sup-lattice*** (*Q-join-complete*) if any $M \in Q^X$ has $\bigsqcup M$ in X.

$\iff$ any $M \in Q^X$ has $\bigsqcap M$ in X ($\bigsqcap M$ defined by analogy to $\bigsqcup M$).

Definition

Let X and Y be sets, and $f : X \to Y$ be a mapping. Zadeh's forward power set operator $f_Q^{\rightarrow} : Q^X \to Q^Y$ maps Q-subsets of X to Q-subsets of Y by

$$f_Q^{\rightarrow}(M)(y) = \bigvee_{x \in f^{-1}(y)} M(x).$$

Note that Zadeh's forward power set operator is a kind of existential quantifier.

For $Q = 2$ we have $y \in f_Q^{\rightarrow}(M) \iff \exists x \in M$ s.t. $f(x) = y$.

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Let (X, e_X) and (Y, e_Y) be Q-ordered sets. A mapping $f\colon (X, e_X) \to (Y, e_Y)$ is *Q-join-preserving* if for any $M \in Q^X$ such that $\bigsqcup_X M$ exists, $\bigsqcup_Y f_Q^{\rightarrow}(M)$ exists and

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Category Q-**Sup**: Q-join-complete sets with Q-join-preserving mappings.

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Category Q-**Sup**: Q-join-complete sets with Q-join-preserving mappings.

Q-**Sup** is isomorphic to Q-**Mod**

Q-sup-lattices are equivalent to Q-modules, where Q is a unital (commutative) quantale. This fact was first pointed out by Stubbe in 2006 and later proved in quantale-like setting by Solovyov in 2016. Thus, Q-modules can be seen as a fuzzification of complete lattices.

Yao introduced L-frames by means of L-ordered sets and proved in 2012 that the categories of Yao-L-frames, Zhang-Liu-L-frames and L-algebras are isomorphic, where L is a frame.

Based on the work of Yao, Wang and Zhao in 2016 independently introduced the notion of Q-quantales, and proved that the category of Q-quantales is isomorphic to the category of Q-algebras.

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1. $F : Q\text{-}\mathbf{Mod} \to Q\text{-}\mathbf{Sup}$, given a Q-module A: $e(a,b) = a \to_Q b$, and $\bigsqcup M = \bigvee_{a \in A}(M(a) * a)$
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3. A mapping that is a morphism in either category, becomes a morphism in the other one as well.
4. $G \circ F = 1_{Q\text{-}\mathbf{Mod}}$ and $F \circ G = 1_{Q\text{-}\mathbf{Sup}}$, i.e., the categories Q-**Mod** and Q-**Sup** are isomorphic.

Results on quantale modules can be directly transferred to Q-sup-lattices. In particular, a nucleus of quantale modules corresponds one-one to a Q-monotone closure operator.

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Outline

Po-algebras and sup-algebras

In 1998 Resende introduced so-called sup-algebras (actually their many-sorted variant).

A *type* is a set Ω of *function symbols*. To each $\omega \in \Omega$, a number $n \in \mathbb{N}_0$ is assigned, which is called the *arity* of ω (and ω is called an n-ary function symbol). Then for each $n \in \mathbb{N}_0$, $\Omega_n \subseteq \Omega$ will denote the subset of all n-ary function symbols from Ω.

Definition

Given a set Ω, an *algebra of type* Ω (shortly, an Ω-*algebra*) is a pair $\mathscr{A} = (A, \Omega)$ where for each $\omega \in \Omega$ with arity n, there is an n-ary operation $f_\omega : A^n \to A$.

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Given a set Ω, an *algebra of type* Ω (shortly, an Ω-*algebra*) is a pair $\mathscr{A} = (A, \Omega)$ where for each $\omega \in \Omega$ with arity n, there is an n-ary operation $f_\omega : A^n \to A$.

Po-algebras and sup-algebras

In 1998 Resende introduced so-called sup-algebras (actually their many-sorted variant).

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Po-algebras and sup-algebras

Definition

A ***partially ordered algebra of type*** Ω (shortly, a ***po-algebra***) is a triple $\mathscr{A} = (A, \leq, \Omega)$ where $(A, \leq)$ is a poset, (A, Ω) is an Ω-algebra, and each operation ω is monotone in any component, that is, $b \leq c$ implies

$$\omega(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n) \leq \omega(a_1, \ldots, a_{j-1}, c, a_{j+1}, \ldots, a_n)$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, $j \in \{1, \ldots, n\}$, and $a_1, \ldots, a_n, b, c \in A$.

Po-algebras and sup-algebras

Definition

A monotone mapping $\phi : A \to B$ from a po-algebra $(A, \leq_A, \Omega)$ to a po-algebra $(B, \leq_B, \Omega)$ is called a

1. ***po-algebra subhomomorphism*** if

$$\omega_B(\phi(a_1), \ldots, \phi(a_n)) \leq_B \phi(\omega_A(a_1, \ldots, a_n))$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, and $a_1, \ldots, a_n \in A$, and, for any $\omega \in \Omega_0$,

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$$\omega\left(a_1, \ldots, a_{j-1}, \bigvee B, a_{j+1}, \ldots, a_n\right) = \bigvee\{\omega(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n) \mid b \in B\}$$

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Po-algebras and sup-algebras

Definition

Sup-algebras of type Ω and their homomorphisms form the category Ω-**SupAlg**.

Instances of sup-algebras that are commonly used include the following (operation arities that are evident from context are omitted):

1. sup-lattices with $\Omega = \emptyset$,
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Q-po-algebras and Q-sup-algebras

In 2016 Šlesinger introduced Q-po-algebras and Q-sup-algebras (one-sorted variant).

Definition

A ***Q-ordered algebra of type*** Ω (shortly, a *Q-ordered algebra*) is a triple $\mathscr{A} = (A, e, \Omega)$ where (A, e) is a Q-ordered set, (A, Ω) is an Ω-algebra, and each operation ω of non-zero arity is Q-monotone in any component, that is,

$$e(b,c) \leq e(\omega(a_1,\ldots,a_{j-1},b,a_{j+1},\ldots,a_n),\omega(a_1,\ldots,a_{j-1},c,a_{j+1},\ldots,a_n))$$

for any $n \in \mathbb{N}$, $\omega \in \Omega_n$, $j \in \{1,\ldots,n\}$, and $a_1,\ldots,a_n,b,c \in A$.

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Q-po-algebras and Q-sup-algebras

Example

Let $(S,\circ)$ be a groupoid, and define multiplication on Q^S by $(A\circ B)(s) = \bigvee_{a\circ b=s}(A(a)\cdot B(b))$. Let $A,B,C\in Q^S$ be arbitrary. Then for any $A,B,C\in Q^S$ and $q\in Q$ we have

$$
\begin{aligned}
q\leq e(B,C) &\iff q\leq \inf_{s\in S}(B(s)\to C(s))\\
&\iff \forall s\in S\colon q\leq B(s)\to C(s) \iff \forall s\in S\colon q\cdot B(s)\leq C(s)\\
&\implies \forall s,c,d\in S \text{ s.t. } c\circ d=s\colon q\cdot A(c)\cdot B(d)\leq A(c)\cdot C(d)\\
&\implies \forall s,c,d\in S \text{ s.t. } c\circ d=s\colon q\cdot (A(c)\cdot B(d))\leq \textstyle\bigvee_{a\circ b=s}(A(a)\cdot C(b))\\
&\iff \forall s\in S\colon q\cdot \textstyle\bigvee_{a\circ b=s}(A(a)\cdot B(b))\leq \bigvee_{a\circ b=s}(A(a)\cdot C(b))\\
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&\implies q\leq \inf_{s\in S}\left(\textstyle\bigvee_{a\circ b=s}(A(a)\cdot B(b))\to \bigvee_{a\circ b=s}(A(a)\cdot C(b))\right)\\
&\qquad = \inf_{s\in S}((A\circ B)(s)\to (A\circ C)(s))\\
&\qquad = e(A\circ B, A\circ C),
\end{aligned}
$$

and the operation $\circ$ is thus Q-monotone on the right-hand side. Verifying Q-monotonicity on the left-hand side of $\circ$ is completely analogous, and $(Q^S,e,\circ)$ is then a Q-ordered groupoid.

Q-po-algebras and Q-sup-algebras

Definition

Let (A, e_A, Ω) and (B, e_B, Ω) be Q-ordered algebras, and $\phi : A \to B$ be a Q-monotone mapping. Then ϕ is called:

1. a Q***-ordered algebra subhomomorphism*** if

$$1 \leq e_B(\omega_B(\phi(a_1), \ldots, \phi(a_n)), \phi(\omega_A(a_1, \ldots, a_n)))$$

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Q-po-algebras and Q-sup-algebras

Definition

A ***Q-sup-algebra of type*** Ω (shortly, a ***Q-sup-algebra***) is a triple $\mathscr{A} = (A, \bigsqcup, \Omega)$ where $(A, \bigsqcup)$ is a Q-sup-lattice, (A, Ω) is an Ω-algebra, and each operation ω is Q-join-preserving in any component, that is,

$$\omega\left(a_1, \ldots, a_{j-1}, \bigsqcup M, a_{j+1}, \ldots, a_n\right) = \bigsqcup \omega(a_1, \ldots, a_{j-1}, -, a_{j+1}, \ldots, a_n)^{\rightarrow}_{Q}(M)$$

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Example

As instances of Q-sup-algebras, we may typically encounter the Q-counterparts of those from examples of sup-algebras:

1. Q-sup-lattices ($\Omega = \emptyset$),
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Q-po-algebras and Q-sup-algebras

Example

Recall that Q^X is the free Q-sup-lattice over a set X. As before, any Ω-algebra A gives rise to a Q-sup-algebra Q^A with operations defined by

$$\omega_{Q^A}(A_1,\ldots,A_n)(a) = \bigvee_{\omega_A(a_1,\ldots,a_n)=a} A_1(a_1)\cdot\cdots\cdot A_n(a_n),$$

given $n \in \mathbb{N}$, $\omega \in \Omega_n$, $A_1,\ldots,A_n \in Q^A$, $a,a_1,\ldots,a_n \in A$.

Definition

Let $(A,\bigsqcup_A,\Omega)$ and $(B,\bigsqcup_B,\Omega)$ be Q-sup algebras, and $\phi : A \to B$ be a Q-join-preserving mapping. Then ϕ is called a ***Q-sup-algebra homomorphism*** if

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Q-po-algebras and Q-sup-algebras

Example

Recall that Q^X is the free Q-sup-lattice over a set X. As before, any Ω-algebra A gives rise to a Q-sup-algebra Q^A with operations defined by

$$\omega_{Q^A}(A_1,\dots,A_n)(a) = \bigvee_{\omega_A(a_1,\dots,a_n)=a} A_1(a_1)\cdot\dots\cdot A_n(a_n),$$

given $n \in \mathbb{N}$, $\omega \in \Omega_n$, $A_1,\dots,A_n \in Q^A$, $a, a_1,\dots,a_n \in A$.

Definition

Let $(A, \bigsqcup_A, \Omega)$ and $(B, \bigsqcup_B, \Omega)$ be Q-sup algebras, and $\phi : A \to B$ be a Q-join-preserving mapping. Then ϕ is called a ***Q-sup-algebra homomorphism*** if

$$\omega_B(\phi(a_1),\dots,\phi(a_n)) = \phi(\omega_A(a_1,\dots,a_n))$$

for any $n \in \mathbb{N}$, $\omega \in \Omega$, and $a_1,\dots,a_n \in A$, and, for any $\omega \in \Omega_0$

$$\omega_B = \phi(\omega_A).$$

Representation theorem for Q-sup-algebras

Definition

Q-sup-algebras of type Ω and their homomorphisms form the category Ω-Q-**SupAlg**.

Like with e.g. quantales, or sup-algebras in general, quotients and subalgebras of Q-sup-algebras can be characterized by means of Q-order nuclei and conuclei acting on the carrier Q-sup-lattice that are also subhomomorphisms of the induced sup-algebras.

For any algebra A of type Ω it can be shown that the set Q^A of all its Q-subsets is a Q-sup-algebra.

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Representation theorem for Q-sup-algebras

Note that there exists the following commutative triangle of the obvious forgetful functors (notice that **Set** is the category of sets and maps and Ω-**Alg** is the category of algebras of type Ω and their homomorphisms:

$$\begin{tikzcd} \Omega\text{-}Q\text{-}\mathbf{SupAlg} \arrow[rr, "U"] \arrow[dr, "V"'] & & \mathbf{Set} \\ & \Omega\text{-}\mathbf{Alg} \arrow[ur, "W"'] & \end{tikzcd}$$

Theorem

The functor $V : \Omega\text{-}Q\text{-}\mathbf{SupAlg} \to \Omega\text{-}\mathbf{Alg}$ has a left adjoint F.

$$F(A) = Q^A$$

Representation theorem for Q-sup-algebras

An analogy of the representation theorem for sup-algebras (in particular for quantales and modules) can then be presented:

Theorem

If $(A, \bigsqcup_A, \Omega)$ is a Q-sup-algebra, then there is a nucleus j on Q^A such that $A \cong (Q^A)_j$.

Representation theorem for Q-sup-algebras

We also have:

Theorem

The category of Q-sup-algebras of type Ω is a monadic construct.

Corollary

The category of Q-sup-algebras of type Ω is complete, cocomplete, wellpowered, extremally co-wellpowered, and has regular factorizations. Moreover, monomorphisms are precisely those morphisms that are injective functions.

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Thank you for your attention.