On Two Approaches to Concrete Dualities and Their Relationships

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Dualities of concrete categories with schizophrenic objects

Let $(\mathbf{C}, |\quad|)$ and $(\mathbf{D}, \|\quad\|)$ be concrete categories over **Set** with $|\quad| : \mathbf{C} \to \mathbf{Set}$ and $\|\quad\| : \mathbf{V} \to \mathbf{Set}.$
Consider $L \in Ob(\mathbf{C})$ and $M \in Ob(\mathbf{D})$.

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Consider $L \in Ob(\mathbf{C})$ and $M \in Ob(\mathbf{D})$.
Every $A \in Ob(\mathbf{C})$ and every $x \in |A|$ determines a map

$$L_{(A,x)} : \mathbf{C}(A, L) \to |L| \text{ by } L_{(A,x)}(h) = |h|(x).$$

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For every $B \in Ob(\mathbf{D})$ and every $y \in \|B\|$,

$$M_{(B,y)} : \mathbf{D}(B, M) \to \|M\| \text{ by } M_{(B,y)}(t) = \|t\|(y).$$

Schizophrenic object

Definition

(L, s, M) is a schizophrenic object iff $s : |L| \to \|M\|$ is a bijection, and

(SO1) For each $A \in Ob(\mathbf{C})$, $\left(\mathbf{C}(A, L) \xrightarrow{s \circ L_{(A,x)}} \|M\|\right)_{x \in |A|}$ has a $\|\ \ \|$-initial lift $\left(T(A) \xrightarrow{T(A)_x} M\right)_{x \in |A|}$,

(SO2) For each $B \in Ob(\mathbf{D})$, $\left(\mathbf{D}(B, M) \xrightarrow{s^{-1} \circ M_{(B,y)}} |L|\right)_{y \in \|M\|}$ has an $|\ \ |$-initial lift $\left(S(B) \xrightarrow{S(B)_y} L\right)_{y \in \|M\|}$.

Theorem

Every schizophrenic object (L, s, M) *induces an adjoint situation (ADS)*

$$(\gamma, \alpha) : S \dashv T : \mathbf{C}^{op} \to \mathbf{D}. \quad (1)$$

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$Fix\,(\alpha)$ = the f.s.c. of $\mathbf{C}$ of all A making α_A an isomorphism in $\mathbf{C}^{op}$,
$Fix\,(\gamma)$ = the f.s.c. of $\mathbf{D}$ of all B making γ_B an isomorphism in $\mathbf{D}$.

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Corollary

The ADS (1) restricts to a duality between $Fix\,(\alpha)$ and $Fix\,(\gamma)$,i.e., $Fix\,(\alpha)^{op} \sim Fix\,(\gamma)$.

Fundamental Duality of Abstract Categories

- **C**: An abstract category with set-indexed products
- $\mathcal{M}$: A class of **C**-monomorphisms
- L: An arbitrarily fixed object of **C**.

Definition

Given a set X,
$\left(X, \tau \overset{m}{\to} L^X\right)$ is a **C**-$\mathcal{M}$-L-space if $\tau \overset{m}{\to} L^X \in \mathcal{M}$.

Definition

The category **C**-$\mathcal{M}$-L-**Top**

Objects: **C**-$\mathcal{M}$-L-spaces.

Morphisms: $\left(X, \tau \overset{m_1}{\to} L^X\right) \overset{f}{\to} \left(Y, \nu \overset{m_2}{\to} L^Y\right)$ s.t. $f : X \to Y$ is a function, and there exists a **C**-morphism $r_f : \nu \to \tau$ making

$$\begin{array}{ccccc} & L^Y & \xrightarrow{f_L^{\leftarrow}} & L^X & \\ m_2 & \uparrow & & \uparrow & m_1 \\ & \nu & \xrightarrow[r_f]{} & \tau & \end{array}$$

commute.

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commute.

$$\begin{array}{ccc} L^Y & \stackrel{f_L^{\leftarrow}}{\longrightarrow} & L^X \\ & \searrow & \downarrow \pi_x \\ & \pi_{f(x)} & \end{array}$$

Theorem

Let $\mathbf{C}$ *be essentially* $(\mathcal{E},\mathcal{M})$*-structured for some* $\mathcal{E} \subseteq Mor(\mathbf{C})$, *i.e.,*
(i) Every $f \in Mor(\mathbf{C})$ *has an* $(\mathcal{E},\mathcal{M})$*-factorization* $f = m \circ e$,
(ii) $\mathbf{C}$ *has a unique* $(\mathcal{E},\mathcal{M})$*-diagonalization property:*
for any $e \in \mathcal{E}$, $m \in \mathcal{M}$ *and* $f, g \in Mor(\mathbf{C})$,

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ f\downarrow & & \downarrow g \\ C & \xrightarrow[m]{} & D \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} A & \xrightarrow{e} & B \\ f\downarrow & \overset{d}{\swarrow} & \downarrow g \\ C & \xrightarrow[m]{} & D \end{array}$$

Then there exists an ADS

$$(\eta,\varepsilon) : L\Omega \dashv LPt : \mathbf{C}^{op} \to \mathbf{C}\text{-}\mathcal{M}\text{-}L\text{-}\mathbf{Top}. \qquad (2)$$

Definition

(i) A $\mathbf{C}$-$\mathcal{M}$-L-space W is L-sober iff $\eta_W \in Iso\,(\mathbf{C}\text{-}\mathcal{M}\text{-}L\text{-}\mathbf{Top})$.
(ii) A $\mathbf{C}$-object A is L-spatial iff $\varepsilon_A \in Iso\,(\mathbf{C}^{op})$.

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$\mathbf{SPA}(\mathbf{C})$=The f.s.c. of $\mathbf{C}$ of all L-spatial objects
$\mathbf{SOBTop}(\mathbf{C})$=The f.s.c. of $\mathbf{C}$-$\mathcal{M}$-L-$\mathbf{Top}$ of all L-sober objects

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Theorem

(FCDT) $\mathbf{SPA}(\mathbf{C})^{op} \sim \mathbf{SOBTop}(\mathbf{C})$.

The relations between two approaches

$(\mathbf{C}, |\quad|)$ and $(\mathbf{D}, \|\quad\|)$ are concrete categories over **Set,** and the category $\mathbf{C}$ is an essentially $(\mathcal{E}, \mathcal{M})$-structured category with set-indexed products for some $\mathcal{E} \subseteq Surj\,(\mathbf{C})$.
The functor $L\Omega : \mathbf{C}\text{-}\mathcal{M}\text{-}L\text{-}\mathbf{Top} \rightarrow \mathbf{C}^{op}$ restricts to

$$L\Omega^{*} : \mathbf{SOBTop}\,(\mathbf{C}) \rightarrow \mathbf{C}^{op},$$

while the functor $LPt : \mathbf{C}^{op} \rightarrow \mathbf{C}\text{-}\mathcal{M}\text{-}L\text{-}\mathbf{Top}$ restricts to

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$$LPt^* : \mathbf{C}^{op} \to \mathbf{SOBTop}\,(\mathbf{C})\,.$$

$$T^* = \mathbf{SOBTop}\,(\mathbf{C}) \xrightarrow{L\Omega^*} \mathbf{C}^{op} \xrightarrow{T} \mathbf{D},$$

$$S^* = \mathbf{D} \xrightarrow{S} \mathbf{C}^{op} \xrightarrow{LPt^*} \mathbf{SOBTop}\,(\mathbf{C})\,.$$

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$S^* \dashv T^* : \mathbf{SOBTop(C)} \longrightarrow \mathbf{D}$.

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Proposition

If $S \dashv T : \mathbf{C}^{op} \to \mathbf{D}$ *is an equivalence, then so is* $S^* \dashv T^* : \mathbf{SOBTop(C)} \longrightarrow \mathbf{D}$.
The converse is also true in case $\mathbf{SPA(C)} = \mathbf{C}$.

Applications to Universal Algebras

A type is a class $\sum$ together with a function *ar* from $\sum$ to the class of all sets. $\sum$ is finitary if the range of *ar* is $\mathbb{N}$

A $\sum$-algebra $\overline{A} = \left(A, (w^A)_{w \in \sum}\right)$ consists of a set A and a family of functions $w^A : A^{ar(w)} \to A$.

A $\sum$-homomorphism from a $\sum$-algebra $\overline{A}$ to another $\sum$-algebra $\overline{B}$ means a function $f : A \to B$ such that

$$\begin{tikzcd} A^{ar(w)} \arrow[r, "f^{ar(w)}"] \arrow[d, "w^A"'] & B^{ar(w)} \arrow[d, "w^B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes, where $f^{ar(w)}(h) = f \circ h$.

$\sum$-algebras and $\sum$-homomorphisms constitute a category $\mathbf{Alg}(\sum)$.

Structured Topological Spaces

Definition

Let $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ be a triple of types. We call a quintuple $\widetilde{X} = \left(X, \left(o^X\right)_{o\in\Delta_1}, \left(p^X\right)_{p\in\Delta_2}, \left(r^X\right)_{r\in\Delta_3}, \tau^X\right)$ a Δ-structured topological space if X is a set, $\left(o^X\right)_{o\in\Delta_1}$ is a family of operations $o^X : X^{ar(o)} \to X$, $\left(p^X\right)_{p\in\Delta_2}$ is a family of partial operations $p^X : dom(p^X) \to X$ with $dom(p^X) \subseteq X^{ar(p)}$, $\left(r^X\right)_{r\in\Delta_3}$ is a family of relations $r^X \subseteq X^{ar(r)}$, and τ^X is a topology on X.

Proposition

*Δ-structured topological spaces form a category Δ-**STOP** with morphisms $f : \widetilde{X} \to \widetilde{Y}$ all functions $f : X \to Y$ satisfying the following conditions:*

(ST1) $f : (X, \tau^X) \to (Y, \tau^Y)$ is continuous,

(ST2) $f : \left(X, (o^X)_{o\in\Delta_1}\right) \to \left(Y, (o^Y)_{o\in\Delta_1}\right)$ is a Δ_1-homomorphism,

(ST3) $f : \left(X, (p^X)_{p\in\Delta_2}\right) \to \left(Y, (p^Y)_{p\in\Delta_2}\right)$ is a Δ_2-partial homomorphism, i.e.,
for each $p \in \Delta_2$ and $h \in dom(p^X)$, $f^{ar(p)}(h) \in dom(p^Y)$ and $p^Y\left(f^{ar(p)}(h)\right) = f\left(p^X(h)\right)$,

(ST4) $f : \left(X, (r^X)_{r\in\Delta_3}\right) \to \left(Y, (r^Y)_{r\in\Delta_3}\right)$ is a Δ_3-relational homomorphism, i.e., for each $r \in \Delta_3$ and $t \in r^X$, $f^{ar(r)}(t) \in r^Y$.

Application of the first approach

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$\mathcal{V}$ and $\mathcal{W}$ can be considered as concrete categories $(\mathcal{V}, |\quad|)$ and $(\mathcal{W}, \|\quad\|)$ with $|\quad| : \mathcal{V} \to \mathbf{Set}$ and $\|\quad\| : \mathcal{W} \to \mathbf{Set}$.

Theorem

Let $\overline{L} = (L, .)$ be a $\mathcal{V}$-object, and $\widetilde{L} = (L, ...)$ a $\mathcal{W}$-object algebraic over $\overline{L}$ in the sense that
(AL1) $o^L : \overline{L}^{ar(o)} \to \overline{L}$ is a $\sum$-homomorphism for each $o \in \Delta_1$,
(AL2) $\overline{dom\,(p^L)}$ is a $\sum$-subalgebra of $\overline{L}^{ar(p)}$, and
$p^L : \overline{dom\,(p^L)} \to \overline{L}$ is a $\sum$-homomorphism for each $p \in \Delta_2$,
*(AL3) $w^L : \widetilde{L}^{ar(w)} \to \widetilde{L}$ is a Δ-**STOP**-morphism.*
Then $L^* = \left(\overline{L}, id_L, \widetilde{L}\right)$ is a schizophrenic object for $(\mathcal{V}, |\quad|)$ and $(\mathcal{W}, \|\quad\|)$, and so induces an adjunction

$$S_{L^*} \dashv T_{L^*} : \mathcal{V}^{op} \to \mathcal{W}.$$

If $\widetilde{L} = (L, ...)$ is algebraic over a $\sum$-algebra $\overline{L}$, then the theorem establishes a dual adjunction between the smallest prevariety in $\mathbf{Alg}(\sum)$ including $\overline{L}$ (denoted $\mathcal{V}\left(\overline{L}\right)$) and the smallest prevariety in Δ-**STOP** including $\widetilde{L}$ (denoted $\mathcal{W}\left(\widetilde{L}\right)$).

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Additionally, if $\sum$ is finatary, L is finite and τ^L is the discrete topology, then this adjunction restricts to a dual adjunction between $\mathcal{V}\left(\overline{L}\right)$ and the f.s.c. of Δ-$\mathbf{STOP}$ of all objects isomorphic to closed substructures of direct powers of $\widetilde{L}$.

Application of the second approach

Definition

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Corollary

Any prevariety $\mathcal{V}$ in $\mathbf{Alg}(\sum)$ including $\overline{L}$ gives rise to an adjunction

$$\overline{L}\Omega \dashv \overline{L}Pt : \mathcal{V}^{op} \to \overline{L}\text{-}\mathbf{Top}.$$

Definition

(i) A $\sum$-algebra $\overline{A}$ is $\overline{L}$-spatial iff for any $a, b \in A$ with $a \neq b$, there exists a $\sum$-homomorphism $h : \overline{A} \to \overline{L}$ such that $h(a) \neq h(b)$.

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(ii) An $\overline{L}$-topological space $(X, \overline{\tau})$ is $\overline{L}$-sober iff for each $\sum$-homomorphism $h : \overline{\tau} \to \overline{L}$, there exists a unique $x \in X$ satisfying $h(\mu) = \mu(x)$ for all $\mu \in \tau$.

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Proposition

For any prevariety $\mathcal{V}$ in $\mathbf{Alg}(\sum)$ *including* $\overline{L}$,
$\mathbf{SPA}(\mathcal{V}) =$ *the f.s.c. of* $\mathcal{V}$ *of all* $\overline{L}$*-spatial objects* $= \mathcal{V}(\overline{L})$
$\mathbf{SOBTop}(\mathcal{V}) \cong$ *the f.s.c. of* $\overline{L}$-$\mathbf{Top}$ *of all* $\overline{L}$*-sober objects* *(*$\overline{L}$-$\mathbf{SobTop}$*).*

Corollary

For any prevariety $\mathcal{V}$ in $\mathbf{Alg}(\sum)$ including $\overline{L}$, and for any prevariety $\mathcal{W}$ in Δ-$\mathbf{STOP}$ including $\widetilde{L}$, if $\widetilde{L}$ is algebraic over $\overline{L}$, then there exists an adjunction

$$S^*_{L^*} \dashv T^*_{L^*} : \overline{L}\text{-}\mathbf{SobTop} \to \mathcal{W}.$$

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