Gelfand duality for compact po-spaces

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At the beginning ...
Extension

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StKSp -- K Haus -- K RF rm
      |      |      |
      ubal  DeV  |
                |
                PrF rm
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StKF rm
A quick reminder

Definition

1. Let $B$ be a ring with an order $\leq$. We say that $B$ is an $\ell$-ring if
   - $(B, \leq)$ is a lattice
   - $a \leq b \Rightarrow a + c \leq b + c$
   - $0 \leq a, b \Rightarrow 0 \leq ab$.

2. An $\ell$-ring $B$ is an $\ell$-algebra if it is an $\mathbb{R}$-algebra such that
   \[(B \ni a \geq 0 \text{ et } \mathbb{R} \ni r \geq 0) \Rightarrow r.a \geq 0.\]

3. A bal is a bounded, archimedean $\ell$-algebra.
   - $a \in B \Rightarrow \exists n \in \mathbb{N} : a \leq n.1$;
   - $(\forall n \in \mathbb{N} \ n.a \leq b) \Rightarrow a \leq 0$. 
A quick reminder

Definition
Let $B$ be a bal.

▶ An $\ell$-ideal $I$ of $B$ is a convex ring-ideal which is $\lor$-closed.
▶ By $\text{Max}(B)$, we denote the set of maximal $\ell$-ideals of $B$.

Lemma
$\text{Max}(B)$ with the topology generated by

$$\omega(b) = \{I \in \text{Max}(B) : I \not\ni b\},$$

is a compact Hausdorff space.
A quick reminder

Definition
Let $X$ be a compact Hausdorff space. By $C(X, \mathbb{R})$, or more simply $C(X)$, we denote the set of continuous functions from $X$ to $\mathbb{R}$.

Lemma
$C(X)$ is a bal.
The uniform norm

Definition
Let $B$ be a bal. For each $b \in B$,

- $|b| = b \lor (-b)$,
- $||b|| = \inf \{\lambda \in \mathbb{R} : |b| \leq \lambda\}$. (the uniform norm)

Theorem (Key observations)

1. $X \cong \text{Max}(C(X))$ for all compact Hausdorff spaces $X$.
2. $B \cong C(\text{Max}(B))$ if and only if $B$ is complete for the uniform norm.
Extension

StKSp → StKFr

K Haus → KRFrm

ubal → DeV

? → PrFr

PrFr →
Shifting the problem
KPSp

Definition
A compact po-space is an ordered compact topological space such that its order is closed.

KPSp is the category of compact po-spaces and increasing continuous functions.

Remark: KHaus could be seen as a full subcategory of KPSp.
Justification

Notation
Let $X$ be a compact po-space.

1. $I(X, \mathbb{R}^+)$, or more simply $I(X)$ denotes the set of increasing continuous functions from $X$ to $\mathbb{R}^+$.

2. $\text{Con}(I(X))$ denotes the set of maximal congruences of $I(X)$.

Theorem
With the right topology on $\text{Con}(I(X))$, it follows

$$X \cong \text{Con}(I(X)).$$
Definition

An \( \ell \text{-semi-ring} \) consists of an algebra \((A, +, ., 0, 1, \leq)\) such that:

- \((A, +, 0)\) and \((A, ., 1)\) are commutative monoids
- \((A, +, .)\) is distributive
- \(a \leq b \iff a + c \leq b + c\)
- \(a \geq 0\)
- \(a \leq b \rightarrow ac \leq bc\)
- \((A, \leq)\) is a lattice.

\(^1\)Note the difference with \(\ell\)-ring.
Definition
An sbal is an ℓ-semi-ring $A$ which is

1. bounded : $a \in A \in A \Rightarrow \exists n \in \mathbb{N} : a \leq n.1$,
2. archimedean : $(\forall n \in \mathbb{N}) n.a + b \leq n.c + d \Rightarrow a \leq c$,
3. an $\mathbb{R}^+$-algebra.

Definition
A morphism between sbals is an application which respects the operations $+, \cdot, \lor, \land, r$ for $r \in \mathbb{R}^+$. 
1. Prove that $sbal$ and $KPSp$ extend $bal$ and $KHaus$.
2. Establish the duality between $sbal$ and $KPSp$.
3. Prove that this duality extends the one between $ubal$ and $KHaus$. 
sbal and bal

Let $A$ be an sbal, we define

$$(a, b) \sim (c, d) \iff a + d = b + c.$$ and

$$A^b = A \times A/ \sim$$

with the operations

- $(a, b) \sim + (c, d) \sim = (a + c, b + d) \sim$
- $(a, b) \sim (c, d) \sim = (ac + bd, ad + bc) \sim$

and the order

- $(a, b) \sim \leq (c, d) \sim \iff a + d \leq c + b.$

**Proposition**

$A^b$ is a bal.
sbal and bal

Proposition
Let $\alpha \in \text{sbal}(A, A')$. Then

$$\alpha^b : A^b \rightarrow (A')^b : (a, b) \sim \mapsto (\alpha(a), \alpha(b)) \sim$$

is a morphism between bals.

Proposition
Let $B, B' \in \text{bal}$ and $\gamma \in \text{bal}(B, B')$. Then

1. $B^+ = \{ b \in B \mid b \geq 0 \} \in \text{sbal}$. 
2. $\text{sbal}(B^+, B'^+) \ni \gamma^+ : B^+ \rightarrow (B')^+ : b \mapsto \gamma(b)$. 
Theorem

Let $B, B' \in \text{bal}$. Then

$$
\eta_B : B \longrightarrow (B^+)^b : b \longmapsto (b^+, b^-) \sim
$$

is an isomorphism such that for each $\gamma \in \text{bal}(B, B')$

$$
\begin{array}{ccc}
B & \xrightarrow{\eta_B} & (B^+)^b \\
\gamma \downarrow & & \downarrow (\gamma^+)^b \\
B' & \xrightarrow{\eta_{B'}} & (B^+)^b.
\end{array}
$$
From the topological side to the algebraic one

Let $F : \text{KPSp} \to \text{sbal}$ be the functor which sends

- a compact po-space $X$ to $I(X)$.
- an increasing continuous function $f : X \to X'$ to

$$\overline{f} : I(X') \to I(X) : g \mapsto g \circ f.$$
Let $A$ be an sbal. We denote $\text{Con}(A)$ the set of the congruences on $A$.

**Proposition**

If $\theta \in \text{Con}(A)$, then

$$\forall a \in A, \exists! s \in \mathbb{R}^+ : (a, s.1) \in \theta.$$  

We denote $\lambda(a, \theta)$ this real.

**Remark** : $(a, b) \in \theta$ iff $\lambda(a, \theta) = \lambda(b, \theta)$.  

From the algebraic side to the topological one

Proposition

1. *With the topology generated by*  
   
   \[ \omega(a, b) = \{ \theta \in \text{Con}_\ell(A) : \theta \not\ni (a, b) \} \]
   
   *\text{Con}_\ell(A) is a compact space.*

2. *The relation \( \triangleleft \) on \text{Con}_\ell(A) defined by*  
   
   \[ \theta \triangleleft \mu \iff \lambda(a, \theta) \leq \lambda(a, \mu) \ \forall a \in A \]
   
   *is a closed order relation.*
From the algebraic side to the topological one

Let $G : \text{sbal} \to \text{KPSp}$ be the functor which sends

- an sbal $A$ to $\text{Con}(A)$.
- an morphism between sbals $\alpha : A \to A'$ to

$$\alpha^* : \text{Con}(A') \to \text{Con}(A) : \theta \mapsto \alpha^{-1}(\theta)$$

where

$$\alpha^{-1}(\theta) = \{(a, b) \in A \mid (\alpha(a), \alpha(b)) \in \theta\}.$$
Theorem

The application

\[ \Phi_A : A \rightarrow \text{I}(\text{Con}(A)) : a \mapsto (f_a : \theta \mapsto \lambda(a, \theta)) \]

is a natural morphism, i.e. for each \( \alpha \in \text{sbal}(A, A') \)

\[
\begin{array}{ccc}
  A & \xrightarrow{\Phi_A} & F(G(A)) \\
  \alpha \downarrow & & \downarrow \alpha^* \\
  A' & \xrightarrow{\Phi_{A'}} & F(G(A')).
\end{array}
\]
Theorem

$\Phi_A$ is a isomorphism if and only if

- $A$ is complete w.r.t. the uniform norm of $A^b$
- $A$ has the "difference with constants" property, i.e. for all $a \in A$ and $r \in \mathbb{R}^+$

$$a \geq r.1 \Rightarrow \exists b \in A : a = r.1 + b.$$ 

We note $\text{usbal}$ the full subcategory of $\text{sbal}$ which contains all $A$ verifying both properties.
Theorem

The application

$$\varepsilon_X : X \rightarrow \text{Con}(I(X)) : x \mapsto \theta_x = \{(f, g) \mid f(x) = g(x)\}$$

is a natural isomorphism, i.e. for each $f \in \text{KPSp}(X, X')$
KPSp $\leftrightarrow$ sbal

**Theorem**

The functors $F$ and $G$ between sbal and KPSp restrict to an equivalence between usbal and KPSp.
Generalization?

- If $X \in KHaus$,
  - $I(X) = C(X)^+$
  - $C(X) \cong (C(X)^+)^b$.

- If $B \in ubal$ then
  - $B^+ \in usbal$
  - $\text{Max}(B) \cong \text{Con}(B^+)$ by
    \[
    I \in \text{Max}(B) \mapsto \theta_I = \{(a, b) \in (B^+)^2 \mid a - b \in I\}.
    \]
Bear