Gelfand duality for compact po-spaces

Laurent De Rudder

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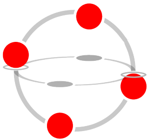


At the beginning ...

$$
\begin{tikzcd}
\textbf{KHaus} \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \textbf{KRFrm} \arrow[d, dash] \\
\textbf{ubal} & \textbf{DeV}
\end{tikzcd}
$$

Extension

$$\begin{tikzcd}
\mathbf{StKSp} \arrow[rrr, dash] & & & \mathbf{StKFrm} \arrow[ddd, dash] \\
& \mathbf{KHaus} \arrow[ul] \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \mathbf{KRFrm} \arrow[ur] \arrow[d, dash] & \\
& \mathbf{ubal} & \mathbf{DeV} \arrow[dr] & \\
& & & \mathbf{PrFrm}
\end{tikzcd}$$

A quick reminder

Definition

1. Let B be a ring with an order $\leq$. We say that B is an ℓ-**ring** if
   - $(B, \leq)$ is a lattice
   - $a \leq b \Rightarrow a + c \leq b + c$
   - $0 \leq a, b \Rightarrow 0 \leq ab$.
2. An ℓ-ring B is an ℓ-**algebra** if it is an $\mathbb{R}$-algebra such that

$$(B \ni a \geq 0 \text{ et } \mathbb{R} \ni r \geq 0) \Rightarrow r.a \geq 0.$$

3. A **bal** is a bounded, archimedean ℓ-algebra.
   - $a \in B \Rightarrow \exists n \in \mathbb{N} \;:\; a \leq n.1$;
   - $(\forall n \in \mathbb{N} \;\; n.a \leq b) \Rightarrow a \leq 0$.

A quick reminder

Definition

Let B be a bal.

- An ℓ-**ideal** I of B is a convex ring-ideal which is $\vee$-closed.
- By $\mathrm{Max}(B)$, we denote the set of maximal ℓ-ideals of B.

Lemma

$\mathrm{Max}(B)$ *with the topology generated by*

$$\omega(b) = \{I \in Max(B) : I \not\ni b\},$$

is a compact Hausdorff space.

A quick reminder

Definition

Let X be a compact Hausdorff space. By $C(X, \mathbb{R})$, or more simply $C(X)$, we denote the set of continuous functions from X to $\mathbb{R}$.

Lemma

$C(X)$ is a bal.

The uniform norm

Definition

Let B be a bal. For each $b \in B$,

- $|b| = b \vee (-b)$,
- $||b|| = \inf\{\lambda \in \mathbb{R} : |b| \leq \lambda.1\}$. **(the uniform norm)**

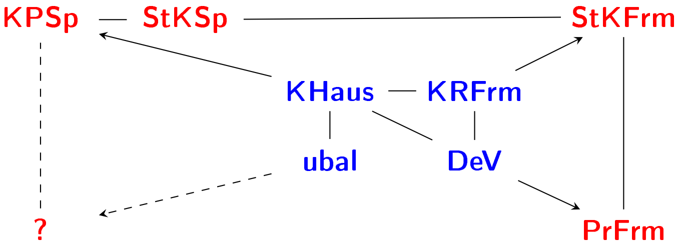
Theorem (Key observations)

1. *$X \cong Max(C(X)$ for all compact Hausdorff spaces X.*
2. *$B \cong C(Max(B))$ if and only if B is complete for the uniform norm.*

Extension

$$\begin{tikzcd}
\mathbf{StKSp} \arrow[rrr, dash] \arrow[ddd, dash, dashed] & & & \mathbf{StKFrm} \arrow[ddd, dash] \\
& \mathbf{KHaus} \arrow[ul] \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \mathbf{KRFrm} \arrow[ur] \arrow[d, dash] & \\
& \mathbf{ubal} \arrow[dl, dashed] & \mathbf{DeV} \arrow[dr] & \\
\mathbf{?} & & & \mathbf{PrFrm}
\end{tikzcd}$$

Shifting the problem



KPSp

Definition

A **compact po-space** is an ordered compact topological space such that its order is closed.

KPSp is the category of compact po-spaces and increasing continuous functions.

Remark : **KHaus** could be seen as a full subcategory of **KPSp**.

Justification

Notation

Let X be a compact po-space.

1. $I(X, \mathbb{R}^+)$, or more simply $I(X)$ denotes the set of increasing continuous functions from X to $\mathbb{R}^+$.
2. $\mathrm{Con}(I(X))$ denotes the set of maximal congruences of $I(X)$.

Theorem

With the right topology on $\mathrm{Con}(I(X))$, *it follows*

$$X \cong \mathrm{Con}(I(X)).$$

Semi-bal

Definition

An ℓ-**semi-ring** consists of an algebra $(A, +, ., 0, 1, \leq)$ such that :

- $(A, +, 0)$ and $(A, ., 1)$ are commutative monoids
- $(A, +, .)$ is distributive
- $a \leq b \Leftrightarrow a + c \leq b + c$ [1]
- $a \geq 0$
- $a \leq b \rightarrow ac \leq bc$
- $(A, \leq)$ is a lattice.

[1] Note the difference with ℓ-ring.

Semi-bal

Definition

An **sbal** is an ℓ-semi-ring A which is

1. bounded : $a \in A \in A \Rightarrow \exists n \in \mathbb{N}: \; a \leq n.1$,
2. archimedean : $(\forall n \in \mathbb{N}) \; n.a + b \leq n.c + d \Rightarrow a \leq c$,
3. an $\mathbb{R}^+$-algebra.

Definition

A morphism between sbals is an application which respects the operations $+, ., \vee, \wedge, r.$ for $r \in \mathbb{R}^+$.

Programme

1. Prove that **sbal** and **KPSp** extend **bal** and **KHaus**.
2. Establish the duality between **sbal** and **KPSp**.
3. Prove that this duality extends the one between **ubal** and **KHaus**.

sbal and **bal**

Let A be an sbal, we define

$$(a,b) \sim (c,d) \Leftrightarrow a+d = b+c.$$

and

$$A^b = A \times A / \sim$$

with the operations

- $(a,b)^\sim + (c,d)^\sim = (a+c, b+d)^\sim$
- $(a,b)^\sim (c,d)^\sim = (ac+bd, ad+bc)^\sim$

and the order

- $(a,b)^\sim \leq (c,d)^\sim \Leftrightarrow a+d \leq c+b.$

Proposition

A^b is a **bal.**

sbal and **bal**

Proposition

Let $\alpha \in \mathbf{sbal}(A, A')$. *Then*

$$\alpha^b : A^b \longrightarrow (A')^b : (a, b)^\sim \longmapsto (\alpha(a), \alpha(b))^\sim$$

is a morphism between bals.

Proposition

Let $B, B' \in \mathbf{bal}$ *and* $\gamma \in \mathbf{bal}(B, B')$. *Then*

1. $B^+ = \{b \in B \mid b \geq 0\} \in \mathbf{sbal}$.
2. $\mathbf{sbal}(B^+, B'^+) \ni \gamma^+ : B^+ \longrightarrow (B')^+ : b \longmapsto \gamma(b)$.

sbal and **bal**

Theorem

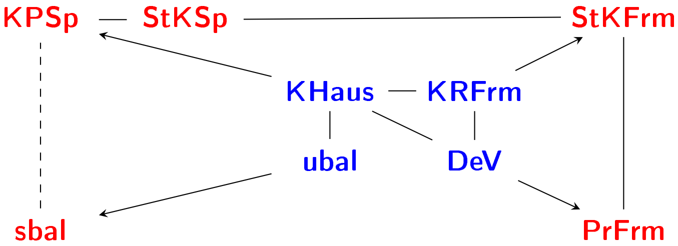
*Let $B, B' \in$ **bal**. Then*

$$\eta_B : B \longrightarrow (B^+)^b : b \longmapsto (b^+, b^-)^\sim$$

*is an isomorphism such that for each $\gamma \in$ **bal**(B, B')*

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[d, "\gamma"'] & (B^+)^b \arrow[d, "(\gamma^+)^b"] \\ B' \arrow[r, "\eta_{B'}"'] & (B^+)^b. \end{tikzcd}$$

Summary



From the topological side to the algebraic one

Let $F : \mathbf{KPSp} \longrightarrow \mathbf{sbal}$ be the functor which sends

- a compact po-space X to $I(X)$.
- an increasing continuous function $f : X \longrightarrow X'$ to

$$\overline{f} : I(X') \longrightarrow I(X) : g \longmapsto g \circ f.$$

From the algebraic side to the topological one

Let A be an sbal. We denote $\mathrm{Con}(A)$ the set of the congruences on A.

Proposition

If $\theta \in \mathrm{Con}(A)$, *then*

$$\forall a \in A,\ \exists! s \in \mathbb{R}^+ :\ (a, s.1) \in \theta.$$

We denote $\lambda(a, \theta)$ *this real.*

Remark : $(a, b) \in \theta$ iff $\lambda(a, \theta) = \lambda(b, \theta)$.

From the algebraic side to the topological one

Proposition

1. *With the topology generated by*

$$\omega(a, b) = \{\theta \in \mathrm{Con}_\ell(A) : \theta \not\ni (a, b)\}$$

   $\mathrm{Con}_\ell(A)$ *is a compact space.*

2. *The relation* $\lhd$ *on* $\mathrm{Con}_\ell(A)$ *defined by*

$$\theta \lhd \mu \Leftrightarrow \lambda(a, \theta) \leq \lambda(a, \mu) \ \forall a \in A$$

   is a closed order relation.

From the algebraic side to the topological one

Let G : **sbal** $\longrightarrow$ **KPSp** be the functor which sends

- an sbal A to $\mathrm{Con}(A)$.
- an moprhism between sbals $\alpha : A \longrightarrow A'$ to

$$\alpha^\star : \mathrm{Con}(A') \longrightarrow \mathrm{Con}(A) : \theta \longmapsto \alpha^{-1}(\theta)$$

where

$$\alpha^{-1}(\theta) = \{(a, b) \in A \mid (\alpha(a), \alpha(b)) \in \theta\}.$$

sbal $\to$ KPSp

Theorem

The application

$$\Phi_A : A \longrightarrow \overbrace{I(\mathrm{Con}(A))}^{=F(G(A))} : a \longmapsto (f_a : \theta \longmapsto \lambda(a,\theta))$$

is a natural morphism, i.e. for each $\alpha \in \textbf{\textit{sbal}}(A, A')$

$$\begin{tikzcd}
A \arrow[r, "\Phi_A"] \arrow[d, "\alpha"'] & F(G(A)) \arrow[d, "\overline{\alpha^\star}"] \\
A' \arrow[r, "\Phi_{A'}"'] & F(G(A')).
\end{tikzcd}$$

Theorem

Φ_A is a isomorphism if and only if

- *A is complete w.r.t. the uniform norm of A^b*
- *A has the "difference with constants" property, i.e. for all $a \in A$ and $r \in \mathbb{R}^+$*

$$a \geq r.1 \Rightarrow \exists b \in A \ : \ a = r.1 + b.$$

We note **usbal** the full subcategory of **sbal** which contains all A verifying both properties.

$\mathbf{KPSp} \to \mathbf{sbal}$

Theorem

The application

$$\varepsilon_X : X \longrightarrow \overbrace{\mathrm{Con}(I(X))}^{=G(F(X))} : x \longmapsto \theta_x = \{(f, g) \mid f(x) = g(x)\}$$

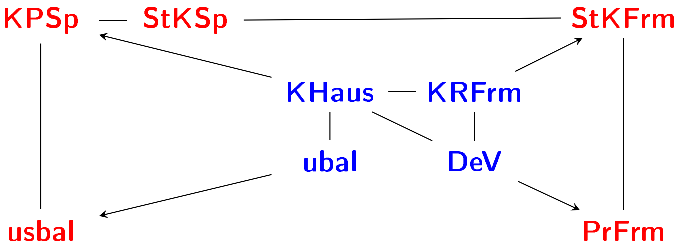
is a natural isomorphism, i.e. for each $f \in \boldsymbol{KPSp}(X, X')$

$$\begin{tikzcd} X \arrow[r, "\varepsilon_X"] \arrow[d, "f"'] & G(F(X)) \arrow[d, "\overline{f}^{\star}"] \\ X' \arrow[r, "\varepsilon_{X'}"'] & G(F(X')). \end{tikzcd}$$

KPSp $\leftrightarrow$ sbal

Theorem

*The functors F and G between **sbal** and **KPSp** restrict to an equivalence between **usbal** and **KPSp**.*


KPSp
StKSp
StKFrm
KHaus
KRFrm
ubal
DeV
usbal
PrFrm

Generalization ?

- If $X \in \mathbf{KHaus}$,
  - $I(X) = C(X)^+$
  - $C(X) \cong (C(X)^+)^b$.
- If $B \in \mathbf{ubal}$ then
  - $B^+ \in \mathbf{usbal}$
  - $Max(B) \cong Con(B^+)$ by

$$I \in Max(B) \longmapsto \theta_I = \{(a, b) \in (B^+)^2 \mid a - b \in I\}.$$

Bear