

A Loomis-Sikorski theorem and functional calculus for a generalized Hermitian algebra

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introduction

GH-algebras – special case of *synaptic algebras* (Foulis, 2010) – unite the notions of an order-unit normed space, a special Jordan algebra, convex effect algebra and an orthomodular lattice.

Examples: self-adjoint part of a von Neumann algebra, AW*-algebra, Rickart C*-algebra; JW-algebra, JB-algebra.

outline

Loomis-Sikorski (LS) theorem

B — Boolean σ -algebra

(X, Σ) — X – Stone space for B ,

Σ — σ -algebra of Baire subsets of X ,

$h : \Sigma \rightarrow B$ — surjective σ -homomorphism.

Generalization for a σ -MV-algebra M Mundici 1999,
Dvurečenskij, 2000:

$(X, \mathcal{T}, h)$, X – Stone space for the maximal Boolean subalgebra of M ,

$\mathcal{T}$ — tribe of functions $f : X \rightarrow [0, 1]$,

$h : \mathcal{T} \rightarrow M$ — surjective σ -homomorphism.

We will present a version of LS-theorem for a commutative GH-algebra and its application in a functional calculus.

synaptic algebra

R – a linear associative algebra with unit element 1.

A – a vector subspace of R , $1 \in A$, partially ordered with positive cone A^+ .

$a, b \in A$ ab taken in R need not belong to A .

aCb means $ab = ba$, the $ab \in A$.

For $B \subseteq A$, $C(B) = \{a \in A : aCb \forall b \in B\}$ - the *commutant* of B .

$C(A)$ -the *center* of A .

$CC(B) = C(C(B))$ - the *double commutant* of $B \subseteq A$.

The vector subspace A of R is a *synaptic algebra* iff the following conditions hold.

axioms of SA

(SA1) With 1 as order-unit, A is an order-unit space with norm $\| \cdot \|$.

(SA2) $a \in A \implies a^2 \in A^+$.

(SA3) $a, b \in A^+ \implies aba \in A^+$.

(SA4) $a \in A, b \in A^+, aba = 0 \implies ab = ba = 0$.

(SA5) $a \in A^+ \implies \exists a^{\frac{1}{2}} \in A^+ \cap CC(a)$ such that $(a^{\frac{1}{2}})^2 = a$.

(SA6) $\forall a \in A \exists a^o \in A$ with $(a^o)^2 = a^o$ and $\forall b \in A$,
 $ab = 0 \iff a^o b = 0$.

(SA7) $1 \leq a \implies \exists a^{-1} \in A$ with $aa^{-1} = a^{-1}a = 1$.

(SA8) If $a, b \in A$, $a_1 \leq a_2 \leq \dots$ are pairwise commuting elements of $C(b)$ and $\lim_{n \rightarrow \infty} \|a - a_n\| = 0$, then $a \in C(b)$.

axioms 1,2,3,4

Assume A is synaptic algebra, non-degenerate, i.e., $0 \neq 1$. So we may identify $\lambda \in \mathbb{R}$ with $\lambda 1$.

- A – order unit space, $A^+ := \{a \in A : 0 \leq a\}$, 1 is an order-unit: $\forall a \in A, \exists \lambda > 0 : a \leq \lambda$.

A is archimedean, i.e. $na \leq b \ \forall n \in \mathbb{N}$ implies $a \leq 0$, with the norm

$$\|a\| := \inf\{0 \leq \lambda \in \mathbb{R} : -\lambda \leq a \leq \lambda\}.$$

- If $a \in A$ then $0 \leq a^2 \in A$, thus A is a special Jordan algebra under the Jordan product

$$a \circ b := \frac{1}{2}((a+b)^2 - a^2 - b^2) = \frac{1}{2}(ab + ba).$$

- $aba = 2a \circ (a \circ b) - a^2 \circ b \in A$; $b \mapsto aba$ is linear and order preserving mapping.

- $a \in A, b \in A^+, aba = 0 \implies ab = ba = 0$; with $b = 1: a^2 = 0 \implies a = 0$.

axioms 5,6,7,8

- $\forall a \in A^+$, *square root* $a^{1/2} \in A^+ \cap CC(a)$ – the unique element such that $(a^{1/2})^2 = a$.
 $|a| := (a^2)^{1/2} \in CC(a)$, $a^+ := \frac{1}{2}(|a| + a)$,
 $a^- := \frac{1}{2}(|a| - a)$, $a = a^+ - a^-$.
- $\forall a \in A$, *carrier projection* $a^o \in CC(a)$ – the unique $p \in A$ such that $p^2 = p$ and, $\forall b \in A$,
 $ab = 0 \Leftrightarrow pb = 0$.
 $|a|^o = a^o$, $(a^n)^o = a^o$, $\forall n \in \mathbb{N}$.
- $a \geq 1 \implies a$ is invertible with $a^{-1} \in CC(a)$; a is invertible iff $|a| \geq \epsilon$, $\epsilon > 0$.
- $\forall a \in A$, $C(a)$ is norm-closed.

effects and projections

- $E := \{e \in A : 0 \leq e \leq 1\}$ is the set of *effects* in A . For $e, f \in E$, define: $e \oplus f$ is defined iff $e + f \in E$, and then $e \oplus f = e + f$.

Then $(E; \oplus, 0, 1)$ is a convex effect algebra with the ordering inherited from A .

- $P := \{p \in A : p^2 = p\}$ is the set of *projections* in A .
- P is an OML, $0 \leq a \leq 1$ for all $a \in P$, orthocomplementation $p \mapsto p^\perp = 1 - p$.

Let $p, q \in P$. Then p and q are

- *orthogonal* ($p \perp q$) iff $p \leq q^\perp$ iff $pq = 0$.
- (Mackey) *compatible* ($p \leftrightarrow q$) iff $p = (p \wedge q) \vee (p \wedge q^\perp)$, $p \leftrightarrow q \iff pCq$.

symmetry, polar decomposition

If $a \in E$, then a^o is the smallest projection $p \in P$ such that $a \leq p$.

$s \in A$ is a *symmetry* iff $s^2 = 1$.

$$-1 \leq s \leq 1, ||s|| = 1.$$

• If s is a symmetry, then $\frac{s+1}{2}$ is a projection.

If p is a projection, then $s = p - p^\perp$ is a symmetry.

• *polar decomposition of a :*

$$a = |a|u = u|a|, |a| = ua = au, u \text{ a symmetry, } u \in CC(a).$$

spectral resolution

- $\forall a \in A$, *spectral resolution*
 $(p_{a,\lambda} : \lambda \in \mathbb{R}) \subseteq CC(a),$

$$p_{a,\lambda} := 1 - ((a - \lambda)^+)^o,$$

$$L := \sup\{\lambda : p_{a,\lambda} = 0\}, U := \inf\{\lambda : p_{a,\lambda} = 1\},$$

$$a = \int_{L-0}^U \lambda dp_{a,\lambda}$$

- $a, b \in A, aCb \Leftrightarrow p_{a,\lambda}Cp_{b,\mu} \forall \lambda, \mu \in \mathbb{R}.$
- The *spectrum* of $a \in A$:
 $\text{spec}(a) := \{\lambda \in \mathbb{R} : a - \lambda \text{ is not invertible}\}.$

GH-algebra

• A synaptic algebra A is a GH-algebra iff
 $a_1 \leq a_2 \leq \dots \leq b, a_n C a_{n+1} \implies \exists a \in A: a_n \nearrow a$.
 Then $a \in CC((a_n)_{n \in \mathbb{N}})$.

• A_1, A_2 – synaptic algebras. A linear mapping
 $\phi : A_1 \rightarrow A_2$ is a *synaptic morphism* iff:

$$\begin{aligned} (1) \quad & \phi(1) = 1. & (2) \quad & \phi(a^2) = \phi(a)^2. \\ (3) \quad & aCb \implies \phi(a)C\phi(b). & (4) \quad & \phi(a^0) = \phi(a)^0. \end{aligned}$$

• A_1, A_2 — GH-algebras. A synaptic morphism
 $\phi : A_1 \rightarrow A_2$ is a *GH-algebra morphism* iff
 $(a_n)_{n \in \mathbb{N}} \subseteq A, a_n C a_{n+1} \ (n \in \mathbb{N}),$

$$(5) \quad a_n \nearrow a \implies \phi(a_n) \nearrow \phi(a)$$

commutative GH-algebra

- A – synaptic algebra. TFAE:
 - (i) A is commutative.
 - (ii) A is lattice ordered, hence an order unit normed vector lattice.
 - (iii) E is an MV-(effect) algebra.
 - (iv) P is a Boolean algebra.
- A commutative synaptic algebra is a GH-algebra iff A is monotone σ -complete.

representation theorem

- A — commutative GH-algebra;
 X — basically disconnected Stone space of the σ -complete Boolean algebra P . Then:
 - (1) $C(X, \mathbb{R})$ is a commutative GH-algebra;
 - (2) $\exists$ GH-isomorphism $\Psi : A \rightarrow C(X, \mathbb{R})$;
 - (3) $\psi := \Psi / P$ to P — Boolean isomorphism of P onto $P(X, \mathbb{R})$ (clopen subsets; Stone's theorem).

states

A – synaptic algebra. A *state* on A – positive linear functional $\rho : A \rightarrow \mathbb{R}$ with $\rho(1) = 1$.

$S(A)$ — state space of A ,

$Ext(S(A))$ — extremal states on A .

• A -commutative GH—algebra,

X —Stone space of P ,

$\Psi : A \rightarrow C(X, \mathbb{R})$ – GH-algebra isomorphism.

• $\rho \in S(A)$ is extremal iff $\rho : A \rightarrow \mathbb{R}$ is a lattice homomorphism iff $\rho(ab) = \rho(a)\rho(b)$, $a, b \in A$.

• There is a bijective correspondence $x \leftrightarrow \rho_x$,
 $x \in X$, $\rho_x \in Ext(S(A))$ such that $\rho_x(a) = \Psi(a)(x)$.
Denote $\hat{a} := \Psi(a)$; then $x(a) = \hat{a}(x) \ \forall x \in X$.

gh-tribe

- A *gh-tribe* is a set $\mathcal{T} \subseteq \mathbb{R}^X$ such that:
 - (1) $\forall f \in \mathcal{T} \exists \alpha, \beta \in \mathbb{R}: \alpha \leq f \leq \beta$.
 - (2) $0, 1 \in \mathcal{T}$.
 - (3) $f, g \in \mathcal{T} \implies f + g \in \mathcal{T}$.
 - (4) $\alpha \in \mathbb{R}, f \in \mathcal{T} \implies \alpha f \in \mathcal{T}$.
 - (5) $(f_n)_{n \in \mathbb{N}}, f \in \mathcal{T}, f_n \leq f, n \in \mathbb{N} \implies \sup_n f_n \in \mathcal{T}$.
- $\mathcal{T}$ is a commutative GH-algebra consisting of bounded functions with pointwise operations.

integral theorem

$\mathcal{T} \subseteq \mathbb{R}^X$ — gh-tribe;

• $\mathcal{B}(\mathcal{T}) := \{D \subseteq X : \chi_D \in \mathcal{T}\}$ — σ -field of sets.

(1) $\forall f \in \mathcal{T}$ is $\mathcal{B}(\mathcal{T})$ -measurable.

(2) $\forall$ σ -additive state ρ on $\mathcal{T}$,

$$\rho(f) = \int_X f(x) \mu(dx),$$

$$\mu(D) := \rho(\chi_D), D \in \mathcal{B}(\mathcal{T}).$$

LS theorem

- *Loomis-Sikorski theorem:*

A — commutative GH-algebra,

X — basically disconnected Stone space of P .

$\implies$

- $\exists$ gh-tribe $(X, \mathcal{T})$ such that $C(X, \mathbb{R}) \subseteq \mathcal{T}$,
- $\exists$ surjective GH-algebra morphism $h : \mathcal{T} \rightarrow A$.

sketch of proof

$\mathcal{T}$ — gh-tribe of functions $f : X \rightarrow \mathbb{R}$ generated by $C(X, \mathbb{R})$.

$X = \text{Ext}S(A)$, $\exists \Psi : A \rightarrow C(X, \mathbb{R})$ — isomorphism,
 $\Psi(a) = \hat{a} \in C(X, \mathbb{R})$, $\hat{a}(x) = x(a)$.

Define $N(f) := \{x \in X : f(x) \neq 0\}$, and $f \sim g$ iff $N(f - g)$ is a meager subset of X , $f, g \in \mathcal{T}$.

Then $\sim$ is an equivalence relation, and $\forall f \in \mathcal{T}$
 $\exists! \hat{a} \in C(X, \mathbb{R})$ with $f \sim \hat{a}$.

Put $h(f) := a$ iff $f \sim \hat{a}$. Then $h : \mathcal{T} \rightarrow A$ is a
GH-algebra morphism.

observables on σ -OMLs

L — σ -OML,

$\mathcal{B}(\mathbb{R})$ — σ -algebra of Borel subsets of $\mathbb{R}$.

• A (real) observable is $\xi : \mathcal{B}(\mathbb{R}) \rightarrow L$ such that

(i) $\xi(\mathbb{R}) = 1$;

(ii) $C \cap D = \emptyset \implies \xi(C \cup D) = \xi(C) \vee \xi(D)$;

(iii) $D_n \nearrow D \implies \xi(D_n) \nearrow \xi(D)$.

ρ — σ -additive state on $L \implies \rho \circ \xi : \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ is the *distribution* of ξ in ρ .

$\rho(\xi) := \int_{\mathbb{R}} \lambda \rho(\xi(d\lambda))$ — the *expectation* of ξ in ρ .

observables on GHA

$(X, \mathcal{T}, h)$ — Loomis-Sikorski representation of $CC(a)$;

$$\forall b \in CC(a), \exists f_b \in \mathcal{T} : h(f_b) = b.$$

Define $\xi_b : \mathcal{B}(\mathbb{R}) \rightarrow P$,

$$\xi_b(D) = h(\chi_{f_b^{-1}(D)}), D \in \mathcal{B}(\mathbb{R}).$$

- ξ_b is a real observable on P , independent on the choice of f_b .
- ξ_b is the unique real observable on P such that

$$\xi_b((-\infty, \lambda]) = p_{b,\lambda}, \lambda \in \mathbb{R}.$$

functional calculus

- ρ — σ -additive state on A ,

$$\rho(a) = \int_{\mathbb{R}} \lambda(\rho(\xi_a(d\lambda))), a \in A.$$

$$f \in C((\text{spec}(a), \mathbb{R}), g := \Psi(a) \in C(X, \mathbb{R}).$$

$$\text{spec}(a) = \{g(x) : x \in X\} \implies f \circ g \in C(X, \mathbb{R}).$$

- Define $f(a) \in CC(a)$ by $f(a) := \Psi^{-1}(f \circ g)$.

$$f(a) = \int_{L_a-0}^{U_a} f(\lambda) dp_{a,\lambda}.$$

$$\rho(f(a)) = \int_{\mathbb{R}} f(\lambda) \rho(\xi_a(d\lambda)).$$

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