Finite MTL-algebras

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Preliminaries

A semihoop is an algebra $\mathbf{A} = (A, \cdot, \rightarrow, \wedge, \vee, 1)$ of type $(2, 2, 2, 2, 0)$ such that $(A, \wedge, \vee)$ is lattice with 1 as greatest element, $(A, \cdot, 1)$ is a commutative monoid and for every $x, y, z \in A$:

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where the subindex i denotes the application of operations in A_i.

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Corollary
For any finite nontrivial MTL-chain M, there are equivalent:

  i. *M is archimedean,*

  ii. *M is simple, and*

  iii *M does not have nontrivial idempotent elements.*

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The category $f\mathcal{LF}$

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Forest Products

Definition

Let $\mathbf{F} = (F, \leq)$ be a forest and let $\{\mathbf{M}_i\}_{i \in \mathbf{F}}$ a collection of MTL-chains such that, up to isomorphism, all they share the same neutral element 1. If $\left(\bigcup_{i \in \mathbf{F}} \mathbf{M}_i\right)^F$ denotes the set of functions $h : F \to \bigcup_{i \in \mathbf{F}} \mathbf{M}_i$ such that $h(i) \in \mathbf{M}_i$ for all $i \in \mathbf{F}$, the forest product $\bigotimes_{i \in \mathbf{F}} \mathbf{M}_i$ is the algebra $\mathbf{M}$ defined as follows:

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Lemma
Let $\mathbf{F}$ be a forest and $\{\mathbf{M}_i\}_{i \in \mathbf{F}}$ a collection of MTL-chains. Then, for every $S \in \mathbb{D}(\mathbf{F})$

$$\mathcal{P}(S) \cong \mathcal{P}(F)/X_S.$$

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Corollary

Let **F** be a forest and $\{\mathbf{M}_i\}_{i \in \mathbf{F}}$ a collection of MTL-chains. Then $\mathcal{P}$ is a sheaf of MTL-chains.

From finite Forest Products to MTL-algebras

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where $\mathcal{P}_m(G) = \bigotimes_{k \in \mathbf{G}} m(k)$, $\mathcal{P}_l(\mathbf{F}) = \bigotimes_{i \in \mathbf{F}} l(i)$ and
$\beta : \mathcal{P}_m(G) \to \bigotimes_{k \in \varphi(F)} m(k)$ is the restriction of $\mathcal{P}_m(G)$ to
$\varphi(\mathbf{F})$.

From finite Forest Products to MTL-algebras

Theorem
The assignments $l \mapsto \mathcal{P}_l(F)$ and $(\varphi, \mathcal{F}) \mapsto \alpha\gamma\beta$ define a contravariant functor

$$\mathcal{H} : f\mathcal{LF} \to f\mathcal{MTL}.$$

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Let us consider the poset of idempotent elements of a MTL-algebra M,

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Corollary

For every finite MTL-algebra M, the poset F_M is a finite forest.

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Lemma

Let M be a finite MTL-algebra and $e \in F_M$. Then $M/\uparrow e$ is archimedean if and only if $e \in m(M)$.

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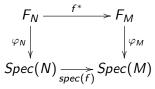*Let M be a finite MTL-algebra, then for every $e \in F_M$, $\uparrow a_e/\uparrow e$ is an archimedean MTL chain.*

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Lemma

Let M and N be finite MTL-algebras and $f : M \to N$ a MTL-algebra morphism. There exists a unique p-morphism $f^* : F_N \to F_M$ such that the diagram

$$
\begin{array}{ccc}
F_N & \xrightarrow{\ f^*\ } & F_M \\
\varphi_N \downarrow & & \downarrow \varphi_M \\
Spec(N) & \xrightarrow[spec(f)]{} & Spec(M)
\end{array}
$$

commutes.

From finite MTL-algebras to Forest Products

Lemma

Let M and N be finite MTL-algebras and $f : M \to N$ be a MTL-algebra morphism. Then, for every $e \in F_N$, f determines a morphism $\dot{f}_e :\uparrow a_{f^*(e)} \to\uparrow a_e$ such that exists a unique MTL-algebra morphism $f_e :\uparrow a_{f^*(e)}/\uparrow f^*(e) \to\uparrow a_e/\uparrow e$ which makes the diagram

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Let M be an id − representable finite MTL-algebra. For every $e \in F_M$, $\uparrow a_e / \uparrow e \cong [a_e, e]$.

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Remark
Let M be an id − representable finite MTL-algebra. For every $e \in F_M$, $\uparrow a_e / \uparrow e \cong [a_e, e]$.

Lemma
For every id − representable finite MTL-algebra M and $m \in Max(F_M)$ it has that $M/\uparrow m \cong \bigoplus_{e \leq m}[a_e, e]$.

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Observe that $F_M = \bigcup_{m \in F_M} \downarrow m$ so the family $\mathcal{R} = \{\downarrow m\}_{m \in M}$ is a covering for F_M.

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Lemma

Let M be an $id - representable$ MTL-algebra. For every $x \in M$, the family $\{f_m(x)\}_{m \in Max(F_M)}$ is a matching family for the covering $\mathcal{R}$.

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Lemma

For every $id-representable$ MTL-algebra M, the assignment $f_M : M \to \mathcal{P}_{I_M}(F_M)$ defined as $f(x) = h_x$, where h_x is the amalgamation of the family $\{f_m(x)\}_{m \in Max(F_M)}$ is an isomorphism.

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- Let $r\mathcal{MTL}$ be the category of *id − representable* finite MTL-algebras.
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- Let us write $\mathcal{G}^*$ for the restriction of the functor $\mathcal{G}$ to the category $r\mathcal{MTL}$ and $\mathcal{H}^*$ for the restriction of the functor $\mathcal{H}$ to $r\mathcal{LF}$.

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- Let $r\mathcal{MTL}$ be the category of *id − representable* finite MTL-algebras.

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Theorem
The categories $r\mathcal{MTL}$ and $r\mathcal{LF}$ are dual.

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Thank you !