A duality for involutive bisemilattices

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Outline

1. Involutive bisemilattices ($\mathcal{IBSL}$).
2. Strongly inverse/direct systems.
3. Płonka sum representation for $\mathcal{IBSL}$.
4. Duality.

Paraconsistent Weak Kleene

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0	0	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$+$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	1	$\frac{1}{2}$	1

$'$	
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- The matrix: $\mathsf{PWK} = \langle \mathbf{WK}, \{1, 1/2\} \rangle$

A closer look to $\mathbf{WK}$

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Let $\mathbf{3}$ be the involution and constants free reduct of $\mathbf{WK}$

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Theorem

$\mathbb{V}(\mathbf{WK}) = \mathcal{IBSL}$.

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I is called the *index set* of the system $\mathcal{X}$, X_i are the *terms* and $p_{ii'}$ are referred to as *bonding morphisms* of $\mathcal{X}$.

Morphisms of strongly inv systems

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A *morphism* between two *strongly inverse systems* $\mathcal{X} = \langle X_i, p_{ii'}, I \rangle$ and $\mathcal{Y} = \langle Y_j, q_{jj'}, J \rangle$, is a pair (φ, f_j) s.t.

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X_{\varphi(j)} \arrow[d, "f_j"'] & & & & X_{\varphi(j')} \arrow[d, "f_{j'}"] \\
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$$\begin{tikzcd} X_{\varphi(j)} \arrow[d, "f_j"'] & X_{\varphi(j')} \arrow[l, "p_{\varphi(j)\varphi(j')}"'] \arrow[d, "f_{j'}"] \\ Y_j & Y_{j'} \arrow[l, "q_{jj'}"] \end{tikzcd}$$

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Duality

Definition

A category $\mathfrak{D}$ is the *dual category* of $\mathfrak{C}$, if there exists an invertible contravariant functor $\mathcal{F} : \mathfrak{C} \to \mathfrak{D}$ with inverse $\mathcal{G}$ s.t. $\mathcal{G} \circ \mathcal{F} = id_{\mathfrak{C}}$ and $\mathcal{G} \circ \mathcal{F} = id_{\mathfrak{D}}$.

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Remark

If $\mathfrak{C}$ and $\mathfrak{D}$ are dual categories, then strong-dir-$\mathfrak{C}$ is the dual category of strong-inv-$\mathfrak{D}$.

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$$g^{\mathcal{P}_l}(a_1, \ldots, a_n) = g^{\mathbf{A}_j}(\varphi_{i_1 j}(a_1), \ldots, \varphi_{i_n j}(a_n)).$$

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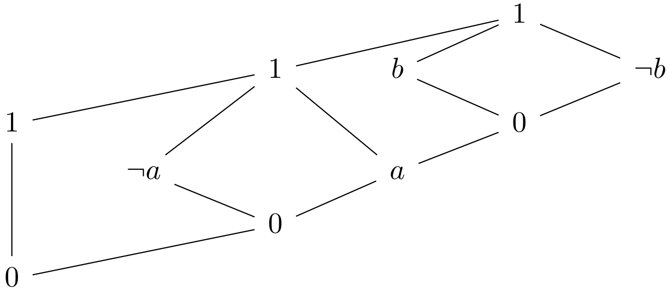
In case ν contains constants, then we define $g = g^{\mathbf{A}_0}$.

Example

$$
\begin{tikzcd}
\mathbf{A}_0 \arrow[r, "\varphi_{01}"] \arrow[rr, bend left, "\varphi_{02}"] & \mathbf{A}_1 \arrow[r, "\varphi_{12}"] & \mathbf{A}_2
\end{tikzcd}
$$



Płonka sums representation

Theorem

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2. *If $\mathbf{B}$ is an involutive bisemilattice, then $\mathbf{B}$ is isomorphic to the Płonka sum over a strongly direct system of Boolean algebras.*

Categories into play

Category	Objects	Morphisms
$\mathfrak{BA}$	Boolean Algebras	Homomorph. of $\mathcal{BA}$
$\mathfrak{IBSL}$	Involutive bisemilattices	Hom. of $\mathcal{IBSL}$
strong-dir-$\mathfrak{BA}$	str. dir. systems of B.A.	Morphisms of s.d.s.
$\mathfrak{SA}$	Stone spaces	continuous maps
strong-inv-$\mathfrak{SA}$	str. inv. systems of Stone sp.	Morphisms of s.i.s.

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Is it possible to describe the dual in terms of a unique space?

Duality for $\mathcal{BSL}$

Theorem (Gierz, Romanowska)

The categories $\mathfrak{DB}$ and $\mathfrak{GR}$ are dual to each other under the functors $\mathrm{Hom}_{\mathrm{b}}(-, \mathbf{3}) : \mathfrak{DB} \to \mathfrak{GR}$ *and* $\mathrm{Hom}_{\mathrm{GR}}(-, \mathbf{3}) : \mathfrak{GR} \to \mathfrak{DB}$.

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2. $\neg(a * b) = \neg a * \neg b$
3. if $a \leq b$ then $\neg b \sqsubseteq \neg a$
4. $\neg c_0 = c_1$, $\neg c_1 = c_0$ and $\neg c_\alpha = c_\alpha$

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5. $\mathrm{Hom}_{\mathrm{GR}}(\mathbf{G}, \mathbf{3})$ with natural involution $\neg$, i.e. $\neg\varphi(a) = (\varphi(\neg a))'$

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5. $\mathrm{Hom}_{\mathrm{GR}}(\mathbf{G}, \mathbf{3})$ with natural involution $\neg$, i.e. $\neg\varphi(a) = (\varphi(\neg a))'$ satisfies $\varphi \cdot (\neg\varphi + \psi) = \psi \cdot \varphi$
6. there exist $\varphi_0, \varphi_1 \in \mathrm{Hom}_{\mathrm{GR}}(\mathbf{G}, \mathbf{3})$ s.t. $\neg\varphi_0 = \varphi_1$ and $\varphi + \varphi_0 = \varphi$, for each $\varphi \in \mathrm{Hom}_{\mathrm{GR}}(\mathbf{G}, \mathbf{3})$.

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Corollary

The category strong-inv-$\mathfrak{SA}$ is equivalent to the category $\mathfrak{IGR}$.

Thank you!