Axiomatizing a Reflexive Real-Valued Modal Logic

Laura Janina Schnüriger

TACL, Praha 29 June 2017

That's the talk about A Real-Valued Modal Logic

That's the talk about

Motivation

Many-valued logics and modal logics are well-established topics

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Motivation

Many-valued logics and modal logics are well-established topics

But the connection is still not a very well-studied field of research

That's the talk about A Real-Valued Modal Logic

A Real-Valued Modal Logic

universe (=truth values): \mathbb{R}

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A Real-Valued Modal Logic

universe (=truth values): \mathbb{R} designated (=true truth values): \mathbb{R}_0^+

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A Real-Valued Modal Logic

universe designated connectives

(=truth values): $\mathbb R$

(=true truth values): \mathbb{R}_0^+

interpreted as arithmetic group operations +, -, 0

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A Real-Valued Modal Logic

universe designated connectives

(=truth values): $\mathbb R$

(=true truth values): \mathbb{R}_0^+

interpreted as arithmetic group operations +, -, 0 plus modality \Box

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Language

 $\mathcal{L} \quad \{\&, \neg, \overline{\mathbf{0}}, \Box\} = \{\rightarrow, \Box\}$

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Language

 $\begin{array}{ll} \mathcal{L} & \{\&, \neg, \overline{0}, \Box\} = \{\rightarrow, \Box\} \\ \hline 0 & = p_0 \rightarrow p_0 & p_0 \in \mathrm{Var} \\ \neg \varphi & = \varphi \rightarrow \overline{0} \\ \varphi \& \psi & = \neg \varphi \rightarrow \psi \\ \Diamond \varphi & = \neg \Box \neg \varphi \end{array}$

That's the talk about A Real-Valued Modal Logic

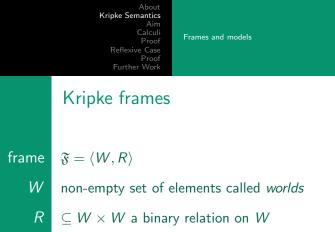
Language

 $\mathcal{L} \quad \{\&, \neg, \overline{0}, \Box\} = \{\rightarrow, \Box\}$ $\overline{0} = p_0 \rightarrow p_0 \qquad p_0 \in \text{Var}$ $\neg \varphi = \varphi \rightarrow \overline{0}$ $\varphi \& \psi = \neg \varphi \rightarrow \psi$ $\Diamond \varphi = \neg \Box \neg \varphi$ $0\varphi = \overline{0}$ $(n+1)\varphi = \varphi \& n\varphi$

Frames and models

Kripke frames

frame $\mathfrak{F} = \langle W, R \rangle$



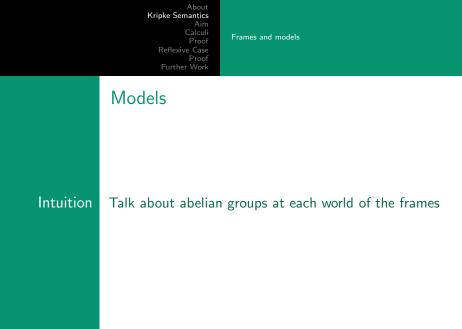
Frames and models

Kripke frames

frame $\mathfrak{F} = \langle W, R \rangle$ W non-empty set of elements called worlds R $\subseteq W \times W$ a binary relation on Wserial a frame \mathfrak{F} is called *serial* if for all $x \in W$ there is $y \in W$ such that Rxy

Kripke frames

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	About Kripke Semantics Aim Calculi Proof Reflexive Case Proof Further Work	Frames and models
	Models	
model	$\mathfrak{M} = \langle W, R, V \rangle$	

	About Kripke Semantics Aim Calculi Proof Reflexive Case Proof Further Work	Frames and models
	Models	
model	$\mathfrak{M} = \langle W, R, V \rangle$	
	$\langle W,R angle$ is a frame	

	About Kripke Semantics Aim Calculi Proof Reflexive Case Proof Further Work
	Models
model	$\mathfrak{M}=\langle W,R,V angle$
	$\langle W,R angle$ is a frame
valuation	$V \colon \operatorname{Var} \times W \to [-r, r]$ for some $r \in \mathbb{R}^+$
	Laura Janina Schnüviger Avienatizing a Deflevive Deal Valued Medal Logic

	About Kripke Semantics Aim Calculi Proof Reflexive Case Proof Further Work	Frames and models
	Models	
model	$\mathfrak{M} = \langle W, R, V angle$	
	$\langle W, R angle$ is a frame	
valuation	$V : \operatorname{Var} \times W \to [-r, r]$ for some $r \in \mathbb{R}^+$ extends to $V : \operatorname{Fm} \times W \to \mathbb{R}$ via	
	$V(\varphi ightarrow \psi; x) = V(\psi; x) - V(\varphi; x)$	
	$V(\Box\varphi;x)=\bigwedge\{V$	$(\varphi; y) \mid Rxy \}$

Frames and models

Models

 $V(\overline{0}; x) = 0$ $V(\neg \varphi; x) = -V(\varphi; x)$ $V(\varphi \& \psi; x) = V(\varphi; x) + V(\psi; x)$ $V(\Diamond \varphi; x) = \bigvee \{V(\varphi; y) \mid Rxy\}$

Frames and models

Models

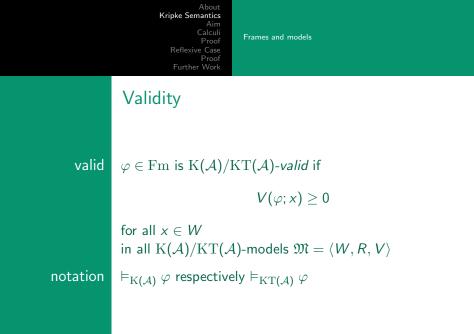
$K(\mathcal{A})$ -model if *R* is serial

Frames and models

Models

 $K(\mathcal{A})$ -modelif R is serial $KT(\mathcal{A})$ -modelif R is reflexive

	About Kripke Semantics Aim Calculi Proof Reflexive Case Proof Further Work		
	Validity		
valid	$arphi \in \mathrm{Fm} ext{ is } \mathrm{K}(\mathcal{A})/\mathrm{KT}(\mathcal{A}) ext{-valid}$ if		
	$V(arphi; x) \geq 0$		
	for all $x \in W$ in all $\mathrm{K}(\mathcal{A})/\mathrm{KT}(\mathcal{A})$ -models $\mathfrak{M} = \langle W, R, V angle$		



Frames and models

Remarks

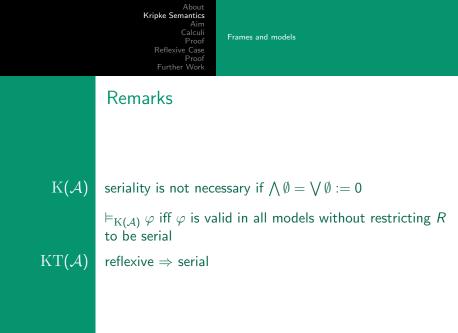
$\mathrm{K}(\mathcal{A})$ seriality is not necessary if $\bigwedge \emptyset = \bigvee \emptyset := 0$

Frames and models

Remarks

 $K(\mathcal{A})$ seriality is not necessary if $\bigwedge \emptyset = \bigvee \emptyset := 0$

 $\vDash_{\mathrm{K}(\mathcal{A})} \varphi \text{ iff } \varphi \text{ is valid in all models without restricting } R$ to be serial





	About Kripke Semantics Aim Calculi Axiom systems Proof Main Theorem 1 Reflexive Case Proof Further Work	
	Aim	
so far	semantical definition of $\mathrm{K}(\mathcal{A})$ and $\mathrm{KT}(\mathcal{A})$	
aim	syntactical description	

	About Kripke Semantics Aim Calculi Proof Reflexive Case Proof Further Work	Axiom systems Main Theorem 1
	Aim	
so far	semantical definition of $\mathrm{K}(\mathcal{A})$ and $\mathrm{KT}(\mathcal{A})$	
aim	syntactical description	
find	axiomatizations \mathfrak{K} φ is derivable in \mathfrak{K} φ is derivable in \mathfrak{K}	

Axiom systems Main Theorem 1

Axiom system \mathfrak{K}

Axiomatization for multiplicative fragment of Abelian logic (logic of abelian groups) ${\cal A}$ plus

Axiom systems Main Theorem 1

Axiom system \Re

Axiomatization for multiplicative fragment of Abelian logic (logic of abelian groups) \mathcal{A} plus axioms (K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$ (D_n) $\Box(n\varphi) \rightarrow n\Box \varphi$ ($n \ge 2$)

Axiom systems Main Theorem 1

Axiom system \mathfrak{K}

Axiomatization for multiplicative fragment of Abelian
logic (logic of abelian groups) \mathcal{A} plusaxioms (K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$
 $\Box(n\varphi) \rightarrow n\Box \varphi \quad (n \ge 2)$ rules (nec) $\varphi/\Box \varphi$
 $n\varphi/\varphi \quad (n \ge 2)$

Axiom systems Main Theorem 1

Axiom system \mathfrak{KT}

 $\begin{array}{ll} \mathfrak{K} \mbox{ plus} \\ \mbox{axiom (T)} & \Box \varphi \rightarrow \varphi \end{array}$

Axiom systems Main Theorem 1

Main Theorem 1

Theorem For any $\varphi \in Fm$:

	About Kripke Semantics Aim Calculi Proof Reflexive Case Proof Further Work	Axiom systems Main Theorem 1	
	Main Theoren	n 1	
Theorem	For any $arphi \in \mathrm{Fm}$:	$\vdash_{\mathfrak{K}} \varphi \Leftrightarrow $	$\vDash_{\mathrm{K}(\mathcal{A})} \varphi$

	About Kripke Semantics Aim Calculi Proof Reflexive Case Proof Further Work	Axiom systems Main Theorem 1			
	Main Theorem	n 1			
Theorem	For any $\varphi \in \operatorname{Fm}$:	$\vdash_{\mathfrak{K}} \varphi$	\Leftrightarrow	$\vDash_{\mathrm{K}(\mathcal{A})} \varphi$	
$Proof \Rightarrow$	easy:				

Axiom systems Main Theorem 1

Main Theorem 1

Theorem Proof \Rightarrow

For any $\varphi \in \operatorname{Fm}$: $\vdash_{\mathfrak{K}} \varphi \iff \vDash_{\operatorname{K}(\mathcal{A})} \varphi$ easy: check that axioms are $\operatorname{K}(\mathcal{A})$ -valid rules preserve validity

Axiom systems Main Theorem 1

Main Theorem 1

Theorem Proof \Rightarrow

For any $\varphi \in \operatorname{Fm}$: $\vdash_{\mathfrak{K}} \varphi \iff \vDash_{\operatorname{K}(\mathcal{A})} \varphi$ easy: check that axioms are $\operatorname{K}(\mathcal{A})$ -valid rules preserve validity not easy:

Axiom systems Main Theorem 1

Main Theorem 1

Theorem Proof \Rightarrow

For any $\varphi \in \operatorname{Fm}$: $\vdash_{\mathfrak{K}} \varphi \iff \vDash_{\operatorname{K}(\mathcal{A})} \varphi$ easy: check that axioms are $\operatorname{K}(\mathcal{A})$ -valid rules preserve validity not easy: via systems $\operatorname{LK}(\mathcal{A})$ and $\operatorname{GK}(\mathcal{A})$

Kripke Semantics Aim Calculi Main Theorem 1 Proof Reflexive Case Proof Further Work Main Theorem 1 Theorem $\vdash_{\mathfrak{K}} \varphi \quad \Leftrightarrow \quad \vDash_{\mathcal{K}(\mathcal{A})} \varphi$ For any $\varphi \in Fm$: $\mathsf{Proof} \Rightarrow$ easy: check that axioms are $K(\mathcal{A})$ -valid rules preserve validity not easy: via systems $LK(\mathcal{A})$ and $GK(\mathcal{A})$ labelled calculus $\vdash_{\mathrm{LK}(\mathcal{A})} \varphi$ sequent calculus $\vdash_{\mathrm{GK}(\mathcal{A})} \Rightarrow \varphi$

About

Kripke Semantics Aim Calculi Main Theorem 1 Proof Reflexive Case Proof Further Work Main Theorem 1 Theorem For any $\varphi \in \operatorname{Fm}$: $\vdash_{\mathfrak{K}} \varphi \iff \vDash_{\mathrm{K}(\mathcal{A})} \varphi$ $\mathsf{Proof} \Rightarrow$ easy: check that axioms are $K(\mathcal{A})$ -valid rules preserve validity not easy: via systems $LK(\mathcal{A})$ and $GK(\mathcal{A})$ labelled calculus $\models_{\mathrm{K}(\mathcal{A})} \varphi \quad \Leftrightarrow \quad \vdash_{\mathrm{LK}(\mathcal{A})} \varphi$ $\vdash_{\mathfrak{K}} \varphi \quad \Leftrightarrow \quad \vdash_{\mathrm{GK}(\mathcal{A})} \Rightarrow \varphi$ sequent calculus

About

About Kripke Semantics Aim Calculi Main Theorem 1 Proof Reflexive Case Proof Further Work Main Theorem 1 Theorem For any $\varphi \in \operatorname{Fm}$: $\vdash_{\mathfrak{K}} \varphi \iff \vDash_{\mathrm{K}(\mathcal{A})} \varphi$ $\mathsf{Proof} \Rightarrow$ easy: check that axioms are $K(\mathcal{A})$ -valid rules preserve validity not easy: via systems $LK(\mathcal{A})$ and $GK(\mathcal{A})$ labelled calculus $\vDash_{\mathrm{K}(\mathcal{A})} \varphi \quad \Leftrightarrow \quad \vdash_{\mathrm{LK}(\mathcal{A})} \varphi$ 1 $\vdash_{\mathfrak{K}} \varphi \quad \Leftrightarrow \quad \vdash_{\mathrm{GK}(\mathcal{A})} \Rightarrow \varphi$ sequent calculus

	About Kripke Semantics Aim Calculi A Semantical Calculus Proof A Syntactical Calculus Reflexive Case Proof Further Work	
	The labelled tableau system $\operatorname{LK}(\mathcal{A})$	
Notation	multisets $\Gamma = [\varphi_1, \dots, \varphi_n], \Delta = [\psi_1, \dots, \psi_m]$ and for $k_1, \dots, k_n, l_1, \dots, l_m \in \mathbb{N}$, let $(\Gamma)^{\mathbf{k}} = [(\varphi_1)^{k_1}, \dots, (\varphi_n)^{k_n}], (\Delta)^{\mathbf{l}} = [(\psi_1)^{l_1}, \dots, (\psi_m)^{l_m}]$	
tableau nodes	are of the forms	
	$(\Gamma)^{\mathbf{k}} \rhd (\Delta)^{\mathbf{l}} \text{ for } \rhd \in \{>, \geq\}$	
•	rij for $i, j \in \mathbb{N}$	

	About Kripke Semantics Aim Calculi Proof Reflexive Case Proof Further Work	A Semantical Calculus A Syntactical Calculus
	The labelled t	ableau system $\mathrm{LK}(\mathcal{A})$
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tableau nodes	are of the forms	
•	$(\Gamma)^{\mathbf{k}} \triangleright (\Delta)^{\mathbf{l}}$ for	
•	<i>rij</i> fo	or $i,j \in \mathbb{N}$
tableau	for a formula φ is a finite sequence of nodes with root	
		$[] > [(\varphi)^1]$ r12

A Semantical Calculus A Syntactical Calculus

The rules

$$\frac{(\Gamma)^{\mathbf{k}},(\psi)^{i} \rhd (\varphi)^{i},(\Delta)^{\mathbf{l}}}{(\Gamma)^{\mathbf{k}},(\varphi \to \psi)^{i} \rhd (\Delta)^{\mathbf{l}}} (\to \rhd) \quad \frac{(\Gamma)^{\mathbf{k}},(\varphi)^{i} \rhd (\psi)^{i},(\Delta)^{\mathbf{l}}}{(\Gamma)^{\mathbf{k}} \rhd (\varphi \to \psi)^{i},(\Delta)^{\mathbf{l}}} (\rhd \to)$$

$$\frac{(\varphi)^{i} \geq (\Box \varphi)^{i}}{(\Gamma)^{k}, (\Box \varphi)^{i} \rhd (\Delta)^{l}} \ (\Box \rhd) \\ rij$$

$$\begin{array}{cc} \operatorname{rij} & (j \in \mathbb{N} \text{ new}) \\ (\Box \varphi)^i \ge (\varphi)^j \\ \overline{(\Gamma)^{\mathsf{k}} \triangleright (\Box \varphi)^i, (\Delta)^{\mathsf{l}}} & (\rhd \Box) \end{array}$$

A Semantical Calculus A Syntactical Calculus

The rules

$$\frac{(\Gamma)^{k},(\psi)^{i} \rhd (\varphi)^{i},(\Delta)^{l}}{(\Gamma)^{k},(\varphi \to \psi)^{i} \rhd (\Delta)^{l}} (\to \rhd) \quad \frac{(\Gamma)^{k},(\varphi)^{i} \rhd (\psi)^{i},(\Delta)^{l}}{(\Gamma)^{k} \rhd (\varphi \to \psi)^{i},(\Delta)^{l}} (\rhd \to)$$

$$\frac{(\varphi)^{j} \ge (\Box \varphi)^{i}}{rij} (\Box \rhd) \qquad \frac{(\Box \varphi)^{i}}{(\Gamma)^{k} (\Box \varphi)^{i} \rhd (\Delta)^{l}} \qquad \frac{(\Box \varphi)^{i}}{(\Gamma)^{k} \rhd}$$

$$\frac{rkj}{rik}$$
 (ex) $(j \in \mathbb{N} \text{ new})$

A Semantical Calculus A Syntactical Calculus

Tableaux

inequations

The system of inequations associated to a tableau consists of all inequations over labelled variables and boxed formulas for each branch, where "," is "+".

A Semantical Calculus A Syntactical Calculus

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closed

a tableau is closed if its associated systems of inequations are all *inconsistent* over $\mathbb R$

A Semantical Calculus A Syntactical Calculus

Tableaux

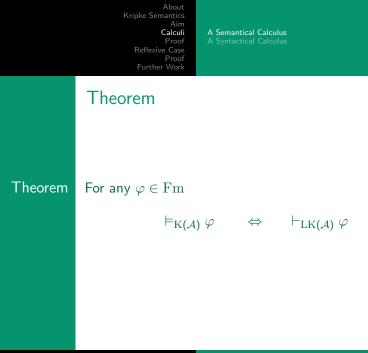
inequations

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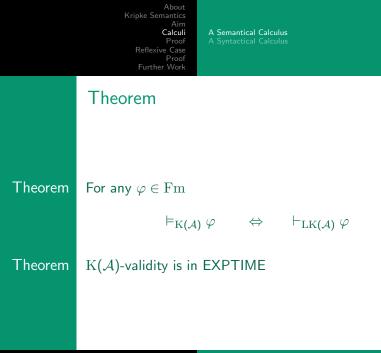
closed

a tableau is closed if its associated systems of inequations are all inconsistent over $\mathbb R$

derivable in $LK(\mathcal{A})$, $\vdash_{LK(\mathcal{A})} \varphi$, if there is a *closed tableau* for φ



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A Semantical Calculus A Syntactical Calculus

The sequent calculus $GK(\mathbb{R})$

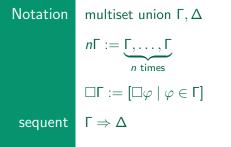
Notation multiset union Γ, Δ

$$n\Gamma := \underbrace{\Gamma, \dots, \Gamma}_{n \text{ times}}$$

 $\Box \mathsf{\Gamma} := [\Box \varphi \mid \varphi \in \mathsf{\Gamma}]$

A Semantical Calculus A Syntactical Calculus

The sequent calculus $GK(\mathbb{R})$



A Semantical Calculus A Syntactical Calculus

The rules

- $\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Pi \Rightarrow \varphi, \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta} \ (CUT)$
- $\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta} \ (\textit{MIX}) \quad \frac{n\Gamma \Rightarrow n\Delta}{\Gamma \Rightarrow \Delta} \ (\textit{sc}_n) \quad (n \ge 2)$
- $\frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} (\to \Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta} (\Rightarrow \to)$

$$\frac{\Gamma \Rightarrow n[\varphi]}{\Box \Gamma \Rightarrow n[\Box \varphi]} \ (\Box_n) \quad (n \ge 0)$$

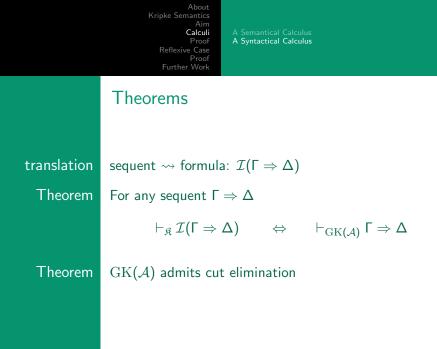
A Semantical Calculus A Syntactical Calculus

Theorems

translation sequent \rightsquigarrow formula: $\mathcal{I}(\Gamma \Rightarrow \Delta)$

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About Kripke Semantics Calculi A Syntactical Calculus Reflexive Case Proof Further Work Theorems translation sequent \rightsquigarrow formula: $\mathcal{I}(\Gamma \Rightarrow \Delta)$ Theorem For any sequent $\Gamma \Rightarrow \Delta$ $\vdash_{\mathfrak{K}} \mathcal{I}(\Gamma \Rightarrow \Delta) \qquad \Leftrightarrow \qquad \vdash_{\mathrm{GK}(\mathcal{A})} \Gamma \Rightarrow \Delta$





To show

$\begin{array}{c|c} \mbox{Theorem 1} & \mbox{For any } \varphi \in \mbox{Fm} : & \vdash_{\mbox{${\rm f}$}} \varphi & \Leftrightarrow & \vDash_{{\rm K}({\mathcal A})} \varphi \end{array}$

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	About Kripke Semantics Aim Calculi Proof Reflexive Case Proof Further Work
	To show
Theorem 1	$For any \ \varphi \in \mathrm{Fm}: \vdash_{\mathfrak{K}} \varphi \Leftrightarrow \ \vDash_{\mathrm{K}(\mathcal{A})} \varphi$
known	$\vDash_{\mathrm{K}(\mathcal{A})} \varphi \Leftrightarrow \vdash_{\mathrm{LK}(\mathcal{A})} \varphi$
	$\vdash_{\mathfrak{K}} \varphi \Leftrightarrow \vdash_{\mathrm{GK}(\mathcal{A})} \Rightarrow \varphi$

	About Kripke Semantics Aim Calculi Proof Reflexive Case Proof Further Work
	To show
Theorem 1	$ \text{For any } \varphi \in \operatorname{Fm}: \vdash_{\mathfrak{K}} \varphi \Leftrightarrow \ \vDash_{\operatorname{K}(\mathcal{A})} \varphi $
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	$\vdash_{\mathfrak{K}} \varphi \Leftrightarrow \vdash_{\mathrm{GK}(\mathcal{A})} \Rightarrow \varphi$
left to show	$ \text{For any } \varphi \in \operatorname{Fm} : \vdash_{\operatorname{LK}(\mathcal{A})} \varphi \Rightarrow \vdash_{\operatorname{GK}(\mathcal{A})} \Rightarrow \varphi $

	About Kripke Semantics Aim Calculi Proof Reflexive Case Proof Further Work
	To show
Theorem 1	
Theorem 1	$ \text{For any } \varphi \in \operatorname{Fm} : \vdash_{\mathfrak{K}} \varphi \Leftrightarrow \ \vDash_{\operatorname{K}(\mathcal{A})} \varphi $
known	$\vDash_{\mathrm{K}(\mathcal{A})} \varphi \Leftrightarrow \vdash_{\mathrm{LK}(\mathcal{A})} \varphi$
	$\vdash_{\mathfrak{K}} \varphi \Leftrightarrow \vdash_{\mathrm{GK}(\mathcal{A})} \Rightarrow \varphi$
left to show	$ \text{For any } \varphi \in \operatorname{Fm} : \vdash_{\operatorname{LK}(\mathcal{A})} \varphi \Rightarrow \vdash_{\operatorname{GK}(\mathcal{A})} \Rightarrow \varphi $
or	$dash_{\mathrm{LK}(\mathcal{A})} \mathcal{I}(\Gamma \Rightarrow \Delta) \Rightarrow dash_{\mathrm{GK}(\mathcal{A})} \Gamma \Rightarrow \Delta$

Proof intuitive idea

induction on sum complexities of $\Gamma \Rightarrow \Delta$

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Proof intuitive idea

induction

on sum complexities of $\Gamma \Rightarrow \Delta$ base case $\Gamma = \Delta$ immediate $\psi \rightarrow \chi \in \Gamma$ or $\psi \rightarrow \chi \in \Delta$ easy

Proof intuitive idea

induction

on sum complexities of $\Gamma \Rightarrow \Delta$ base case $\Gamma = \Delta$ immediate $\psi \rightarrow \chi \in \Gamma$ or $\psi \rightarrow \chi \in \Delta$ easy interesting case: $\Box \Gamma' \Rightarrow \Box \Delta'$

Proof intuitive idea

 $\vdash_{\mathrm{LK}(\mathcal{A})} \mathcal{I}(\Box \Gamma' \Rightarrow \Box \Delta')$

Proof intuitive idea

 $\vdash_{\mathrm{LK}(\mathcal{A})} \mathcal{I}(\Box \Gamma' \Rightarrow \Box \Delta')$

there is a tableau with inconsistent set of inequations

Proof intuitive idea

 $\vdash_{\mathrm{LK}(\mathcal{A})} \mathcal{I}(\Box \mathsf{\Gamma}' \Rightarrow \Box \Delta')$

there is a tableau with inconsistent set of inequations

linear programming problem over ${\mathbb R}$

Proof intuitive idea $\vdash_{LK(\mathcal{A})} \mathcal{I}(\Box \Gamma' \Rightarrow \Box \Delta')$ \Rightarrow there is a tableau with inconsistent set of inequations \Rightarrow linear programming problem over \mathbb{R} \Rightarrow gives formulas $\mathcal{I}(\Sigma_i \Rightarrow \Pi_i)$ derivable in LK(\mathcal{A}) for some $i = 1, \dots, m$

Proof intuitive idea $\vdash_{\mathrm{LK}(\mathcal{A})} \mathcal{I}(\Box \mathsf{\Gamma}' \Rightarrow \Box \Delta')$ there is a tableau with inconsistent set of inequations linear programming problem over \mathbb{R} gives formulas $\mathcal{I}(\Sigma_i \Rightarrow \Pi_i)$ derivable in LK(\mathcal{A}) for some i = 1, ..., mapply induction hypothesis

Proof intuitive idea $\vdash_{\mathrm{LK}(\mathcal{A})} \mathcal{I}(\Box \mathsf{\Gamma}' \Rightarrow \Box \Delta')$ there is a tableau with inconsistent set of inequations linear programming problem over \mathbb{R} gives formulas $\mathcal{I}(\Sigma_i \Rightarrow \Pi_i)$ derivable in LK(\mathcal{A}) for some i = 1, ..., mapply induction hypothesis $\vdash_{\mathrm{GK}(\mathcal{A})} \Sigma_i \Rightarrow \prod_i$

Proof intuitive idea $\vdash_{\mathrm{LK}(\mathcal{A})} \mathcal{I}(\Box \mathsf{\Gamma}' \Rightarrow \Box \Delta')$ there is a tableau with inconsistent set of inequations linear programming problem over \mathbb{R} gives formulas $\mathcal{I}(\Sigma_i \Rightarrow \Pi_i)$ derivable in LK(A) for some $i = 1, \ldots, m$ apply induction hypothesis $\vdash_{\mathrm{GK}(\mathcal{A})} \Sigma_i \Rightarrow \prod_i$ apply $(sc_k), (MIX)$ and (\Box_n) to get $\vdash_{\mathrm{GK}(\mathbb{R})} \Box \Gamma' \Rightarrow \Box \Delta'$

Main Theorem 2 Calculi

Main Theorem 2

Recall \mathfrak{KT} is \mathfrak{K} plus $\Box \varphi \to \varphi$

Main Theorem 2 Calculi

Main Theorem 2

Recall \mathfrak{KT} is \mathfrak{K} plus $\Box \varphi \rightarrow \varphi$ Theorem 2For any $\varphi \in \mathrm{Fm}$: $\vdash_{\mathfrak{KT}} \varphi \Leftrightarrow \vDash_{\mathrm{KT}(\mathcal{A})} \varphi$

Main Theorem 2 Reflexive Case Proof Further Work Main Theorem 2 Recall \mathfrak{KT} is \mathfrak{K} plus $\Box \varphi \to \varphi$ Theorem 2 $\text{For any } \varphi \in \operatorname{Fm} : \qquad \vdash_{\mathfrak{KT}} \varphi \quad \Leftrightarrow \ \models_{\operatorname{KT}(\mathcal{A})} \varphi$ Proof same idea as for $K(\mathcal{A})$

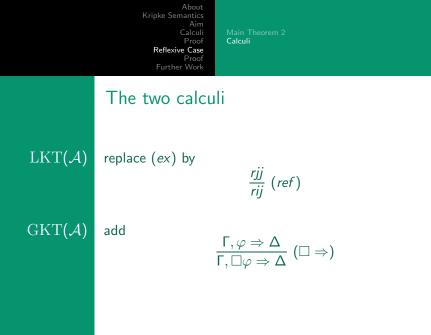
About Kripke Semantics

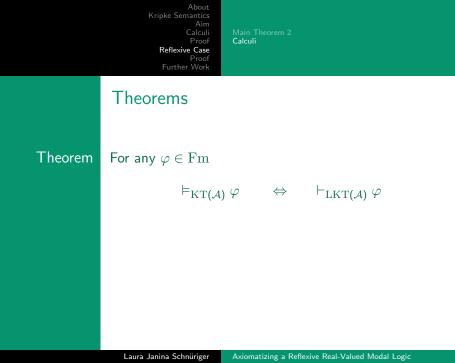
Main Theorem 2 Calculi

The two calculi

 $LKT(\mathcal{A})$ replace (*ex*) by

 $\frac{rjj}{rij}$ (ref)





	About Kripke Semantics Aim Calculi Main Theorem 2 Proof Reflexive Case Proof Further Work
	Theorems
Theorem	For any $arphi \in \mathrm{Fm}$
	$\vDash_{\mathrm{KT}(\mathcal{A})} \varphi \Leftrightarrow \vdash_{\mathrm{LKT}(\mathcal{A})} \varphi$
Theorem	For any $arphi \in \mathrm{Fm}$
	$\vDash_{\mathfrak{K}\mathfrak{T}}\mathcal{I}(\Gamma\Rightarrow\Delta)\qquad\Leftrightarrow\qquad \vdash_{\mathrm{GKT}(\mathcal{A})}\Gamma\Rightarrow\Delta$

What's different to before?

before \Box 's were removed simultaneously on the left and the right hand side of a sequent

What's different to before?

before \Box 's were removed simultaneously on the left and the right hand side of a sequent

now \Box can be removed just on left hand side by $(\Box \Rightarrow)$

What's different to before?

- before \Box 's were removed simultaneously on the left and the right hand side of a sequent
 - now \Box can be removed just on left hand side by $(\Box \Rightarrow)$
 - it is more complicated to apply induction hypothesis

Further Work

Find axiomatizations for transitive (and symmetric) case

Further Work

Find axiomatizations for transitive (and symmetric) case
 add lattice operators ∧ and ∨

Further Work

- Find axiomatizations for transitive (and symmetric) case
 add lattice operators ∧ and ∨
- Find complexitiy class of decidability for K(A) (is it EXPTIME-complete?)

Kripke Semantics Aim Calculi Appendix Proof Reflexive Case	Proof Further Work
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References

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Appendix

Axiom system $K(\mathcal{A})$

axioms

rules

(B) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ (C) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$ (I) $\varphi \to \varphi$ (A) $((\varphi \to \psi) \to \psi) \to \varphi$ (K) $\Box(\varphi \to \psi) \to (\Box \varphi \to \Box \psi)$ (D_n) $\Box(n\varphi) \rightarrow n\Box\varphi$ $(n \ge 2)$ (mp) $\{\varphi, \varphi \to \psi\}/\psi$ (nec) $\varphi / \Box \varphi$ (con_n) $n\varphi/\varphi$ (n > 2)