Axiomatizing a Reflexive Real-Valued Modal Logic

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That's the talk about

Motivation

Many-valued logics and modal logics are well-established topics

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Many-valued logics and modal logics are well-established topics

But the connection is still not a very well-studied field of research

A Real-Valued Modal Logic

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plus modality $\Box$

Language

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$$\mathcal{L} \quad \{\&, \neg, \overline{0}, \Box\} = \{\to, \Box\}$$

$$\overline{0} = p_0 \to p_0 \qquad p_0 \in \mathrm{Var}$$

$$\neg\varphi = \varphi \to \overline{0}$$

$$\varphi \& \psi = \neg\varphi \to \psi$$

$$\Diamond\varphi = \neg\Box\neg\varphi$$

Language

$$\mathcal{L} \quad \{\&, \neg, \overline{0}, \Box\} = \{\rightarrow, \Box\}$$

$$\begin{aligned}
\overline{0} &= p_0 \rightarrow p_0 \qquad\qquad p_0 \in \mathrm{Var} \\
\neg\varphi &= \varphi \rightarrow \overline{0} \\
\varphi\&\psi &= \neg\varphi \rightarrow \psi \\
\Diamond\varphi &= \neg\Box\neg\varphi \\
0\varphi &= \overline{0} \\
(n+1)\varphi &= \varphi\& n\varphi
\end{aligned}$$

Kripke frames

frame $\mathfrak{F} = \langle W, R \rangle$

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R	$\subseteq W \times W$ a binary relation on W
serial	a frame $\mathfrak{F}$ is called *serial* if for all $x \in W$ there is $y \in W$ such that Rxy
reflexive	if for all $x \in W$: Rxx

Models

Intuition

Talk about abelian groups at each world of the frames

Models

model $\mathfrak{M} = \langle W, R, V \rangle$

Models

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Models

model $\mathfrak{M} = \langle W, R, V \rangle$

$\langle W, R \rangle$ is a frame

valuation $V \colon \mathrm{Var} \times W \to [-r, r]$ for some $r \in \mathbb{R}^+$
extends to $V \colon \mathrm{Fm} \times W \to \mathbb{R}$ via

$$V(\varphi \to \psi; x) = V(\psi; x) - V(\varphi; x)$$

$$V(\Box\varphi; x) = \bigwedge\{V(\varphi; y) \mid Rxy\}$$

Models

$$V(\overline{0}; x) = 0$$

$$V(\neg\varphi; x) = -V(\varphi; x)$$

$$V(\varphi \& \psi; x) = V(\varphi; x) + V(\psi; x)$$

$$V(\Diamond\varphi; x) = \bigvee\{V(\varphi; y) \mid Rxy\}$$

Models

$\mathrm{K}(\mathcal{A})$-model if R is serial

Models

$\mathrm{K}(\mathcal{A})$-model	if R is serial
$\mathrm{KT}(\mathcal{A})$-model	if R is reflexive

Validity

valid

$\varphi \in \mathrm{Fm}$ is $\mathrm{K}(\mathcal{A})/\mathrm{KT}(\mathcal{A})$-*valid* if

$$V(\varphi; x) \geq 0$$

for all $x \in W$
in all $\mathrm{K}(\mathcal{A})/\mathrm{KT}(\mathcal{A})$-models $\mathfrak{M} = \langle W, R, V\rangle$

Validity

valid $\varphi \in \mathrm{Fm}$ is $\mathrm{K}(\mathcal{A})/\mathrm{KT}(\mathcal{A})$-*valid* if

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notation $\models_{\mathrm{K}(\mathcal{A})} \varphi$ respectively $\models_{\mathrm{KT}(\mathcal{A})} \varphi$

Remarks

$\mathrm{K}(\mathcal{A})$ seriality is not necessary if $\bigwedge \emptyset = \bigvee \emptyset := 0$

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$\mathrm{K}(\mathcal{A})$

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$\models_{\mathrm{K}(\mathcal{A})} \varphi$ iff φ is valid in all models without restricting R to be serial

$\mathrm{KT}(\mathcal{A})$ reflexive $\Rightarrow$ serial

Aim

so far semantical definition of $\mathrm{K}(\mathcal{A})$ and $\mathrm{KT}(\mathcal{A})$

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aim syntactical description

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find axiomatizations $\mathfrak{K}$ and $\mathfrak{KT}$ such that

φ is derivable in $\mathfrak{K} \quad \Leftrightarrow \quad \models_{\mathrm{K}(\mathcal{A})} \varphi$

φ is derivable in $\mathfrak{KT} \quad \Leftrightarrow \quad \models_{\mathrm{KT}(\mathcal{A})} \varphi$

Axiom system $\mathfrak{K}$

Axiomatization for multiplicative fragment of Abelian logic (logic of abelian groups) $\mathcal{A}$ plus

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axioms (K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

$(\mathrm{D_n})$ $\Box(n\varphi) \to n\Box\varphi \qquad (n \geq 2)$

Axiom system $\mathfrak{K}$

Axiomatization for multiplicative fragment of Abelian logic (logic of abelian groups) $\mathcal{A}$ plus

axioms (K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$(\mathrm{D_n})$	$\Box(n\varphi) \to n\Box\varphi \qquad (n \geq 2)$
rules (nec)	$\varphi / \Box\varphi$
$(\mathrm{con_n})$	$n\varphi / \varphi \qquad (n \geq 2)$

Axiom system $\mathfrak{KT}$

$\mathfrak{K}$ plus

axiom (T) $\Box\varphi \to \varphi$

Main Theorem 1

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$\vdash_{\mathrm{LK}(\mathcal{A})} \varphi$ labelled calculus

$\vdash_{\mathrm{GK}(\mathcal{A})} \Rightarrow \varphi$ sequent calculus

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$$\begin{array}{ccc} \models_{\mathrm{K}(\mathcal{A})} \varphi & \Leftrightarrow & \vdash_{\mathrm{LK}(\mathcal{A})} \varphi \\ & & \Downarrow \\ \vdash_{\mathfrak{K}} \varphi & \Leftrightarrow & \vdash_{\mathrm{GK}(\mathcal{A})} \Rightarrow \varphi \end{array}$$

labelled calculus

sequent calculus

The labelled tableau system $\mathrm{LK}(\mathcal{A})$

Notation

multisets $\Gamma = [\varphi_1, \ldots, \varphi_n], \Delta = [\psi_1, \ldots, \psi_m]$
and for $k_1, \ldots, k_n, l_1, \ldots, l_m \in \mathbb{N}$, let
$(\Gamma)^{\mathbf{k}} = [(\varphi_1)^{k_1}, \ldots, (\varphi_n)^{k_n}], (\Delta)^{\mathbf{l}} = [(\psi_1)^{l_1}, \ldots, (\psi_m)^{l_m}]$

tableau nodes

are of the forms

- $(\Gamma)^{\mathbf{k}} \rhd (\Delta)^{\mathbf{l}}$ for $\rhd \in \{>, \geq\}$
- rij for $i, j \in \mathbb{N}$

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tableau nodes are of the forms

- $(\Gamma)^{\mathbf{k}} \rhd (\Delta)^{\mathbf{l}}$ for $\rhd \in \{>, \geq\}$
- rij for $i, j \in \mathbb{N}$

tableau for a formula φ is a finite sequence of nodes with *root*

$$[] > [(\varphi)^1]$$
$$r12$$

The rules

$$\frac{(\Gamma)^{\mathsf{k}}, (\psi)^{i} \rhd (\varphi)^{i}, (\Delta)^{\mathsf{l}}}{(\Gamma)^{\mathsf{k}}, (\varphi \to \psi)^{i} \rhd (\Delta)^{\mathsf{l}}} \ (\to\rhd) \qquad \frac{(\Gamma)^{\mathsf{k}}, (\varphi)^{i} \rhd (\psi)^{i}, (\Delta)^{\mathsf{l}}}{(\Gamma)^{\mathsf{k}} \rhd (\varphi \to \psi)^{i}, (\Delta)^{\mathsf{l}}} \ (\rhd\to)$$

$$\frac{(\varphi)^{j} \geq (\Box\varphi)^{i}}{\underset{rij}{(\Gamma)^{\mathsf{k}}, (\Box\varphi)^{i} \rhd (\Delta)^{\mathsf{l}}}} \ (\Box \rhd) \qquad \frac{\overset{rij \qquad (j \in \mathbb{N} \text{ new})}{(\Box\varphi)^{i} \geq (\varphi)^{j}}}{(\Gamma)^{\mathsf{k}} \rhd (\Box\varphi)^{i}, (\Delta)^{\mathsf{l}}} \ (\rhd \Box)$$

The rules

$$\frac{(\Gamma)^{\mathsf{k}}, (\psi)^{i} \rhd (\varphi)^{i}, (\Delta)^{\mathsf{l}}}{(\Gamma)^{\mathsf{k}}, (\varphi \to \psi)^{i} \rhd (\Delta)^{\mathsf{l}}} \ (\to\rhd) \qquad \frac{(\Gamma)^{\mathsf{k}}, (\varphi)^{i} \rhd (\psi)^{i}, (\Delta)^{\mathsf{l}}}{(\Gamma)^{\mathsf{k}} \rhd (\varphi \to \psi)^{i}, (\Delta)^{\mathsf{l}}} \ (\rhd\to)$$

$$\frac{(\varphi)^{j} \geq (\Box\varphi)^{i}}{\begin{array}{c}(\Gamma)^{\mathsf{k}}, (\Box\varphi)^{i} \rhd (\Delta)^{\mathsf{l}} \\ rij\end{array}} \ (\Box \rhd) \qquad \frac{\begin{array}{c} rij \\ (\Box\varphi)^{i} \geq (\varphi)^{j}\end{array}}{(\Gamma)^{\mathsf{k}} \rhd (\Box\varphi)^{i}, (\Delta)^{\mathsf{l}}} \ (\rhd \Box) \quad (j \in \mathbb{N} \text{ new})$$

$$\frac{rkj}{rik} \ (ex) \quad (j \in \mathbb{N} \text{ new})$$

Tableaux

inequations

The *system of inequations* associated to a tableau consists of all inequations over labelled variables and boxed formulas for each branch, where "," is "$+$".

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closed a tableau is closed if its associated systems of inequations are all *inconsistent* over $\mathbb{R}$

derivable in $\mathrm{LK}(\mathcal{A})$, $\vdash_{\mathrm{LK}(\mathcal{A})} \varphi$, if there is a *closed tableau* for φ

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Theorem $\mathrm{K}(\mathcal{A})$-validity is in EXPTIME

The sequent calculus $\mathrm{GK}(\mathbb{R})$

Notation

multiset union Γ, Δ

$n\Gamma := \underbrace{\Gamma, \ldots, \Gamma}_{n \text{ times}}$

$\Box\Gamma := [\Box\varphi \mid \varphi \in \Gamma]$

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sequent

$\Gamma \Rightarrow \Delta$

The rules

$$\frac{}{\Delta \Rightarrow \Delta}\ (ID) \qquad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Pi \Rightarrow \varphi, \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (CUT)$$

$$\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (MIX) \qquad \frac{n\Gamma \Rightarrow n\Delta}{\Gamma \Rightarrow \Delta}\ (sc_n) \quad (n \geq 2)$$

$$\frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)$$

$$\frac{\Gamma \Rightarrow n[\varphi]}{\Box\Gamma \Rightarrow n[\Box\varphi]}\ (\Box_n) \quad (n \geq 0)$$

Theorems

translation

sequent $\rightsquigarrow$ formula: $\mathcal{I}(\Gamma \Rightarrow \Delta)$

Theorems

translation sequent $\rightsquigarrow$ formula: $\mathcal{I}(\Gamma \Rightarrow \Delta)$

Theorem For any sequent $\Gamma \Rightarrow \Delta$

$$\vdash_{\mathfrak{K}} \mathcal{I}(\Gamma \Rightarrow \Delta) \qquad \Leftrightarrow \qquad \vdash_{\mathrm{GK}(\mathcal{A})} \Gamma \Rightarrow \Delta$$

Theorems

translation: sequent $\rightsquigarrow$ formula: $\mathcal{I}(\Gamma \Rightarrow \Delta)$

Theorem: For any sequent $\Gamma \Rightarrow \Delta$

$$\vdash_{\mathfrak{K}} \mathcal{I}(\Gamma \Rightarrow \Delta) \quad \Leftrightarrow \quad \vdash_{\mathrm{GK}(\mathcal{A})} \Gamma \Rightarrow \Delta$$

Theorem: $\mathrm{GK}(\mathcal{A})$ admits cut elimination

To show

Theorem 1 For any $\varphi \in \mathrm{Fm}$: $\quad \vdash_{\mathfrak{K}} \varphi \quad \Leftrightarrow \quad \models_{\mathrm{K}(\mathcal{A})} \varphi$

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left to show For any $\varphi \in \mathrm{Fm}$: $\quad \vdash_{\mathrm{LK}(\mathcal{A})} \varphi \quad \Rightarrow \quad \vdash_{\mathrm{GK}(\mathcal{A})} \Rightarrow \varphi$

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left to show For any $\varphi \in \mathrm{Fm}$: $\quad \vdash_{\mathrm{LK}(\mathcal{A})} \varphi \quad \Rightarrow \quad \vdash_{\mathrm{GK}(\mathcal{A})} \Rightarrow \varphi$

or $\vdash_{\mathrm{LK}(\mathcal{A})} \mathcal{I}(\Gamma \Rightarrow \Delta) \quad \Rightarrow \quad \vdash_{\mathrm{GK}(\mathcal{A})} \Gamma \Rightarrow \Delta$

Proof intuitive idea

induction on sum complexities of $\Gamma \Rightarrow \Delta$

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$\psi \to \chi \in \Gamma$ or $\psi \to \chi \in \Delta$ easy

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interesting case: $\Box\Gamma' \Rightarrow \Box\Delta'$

Proof intuitive idea

$\vdash_{\mathrm{LK}(\mathcal{A})} \mathcal{I}(\Box\Gamma' \Rightarrow \Box\Delta')$

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$\Rightarrow$ there is a tableau with inconsistent set of inequations

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$\Rightarrow$ linear programming problem over $\mathbb{R}$

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$\vdash_{\mathrm{LK}(\mathcal{A})} \mathcal{I}(\Box\Gamma' \Rightarrow \Box\Delta')$

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$\Rightarrow$ gives formulas $\mathcal{I}(\Sigma_i \Rightarrow \Pi_i)$ derivable in $\mathrm{LK}(\mathcal{A})$ for some $i = 1, \ldots, m$

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$\Rightarrow$ linear programming problem over $\mathbb{R}$

$\Rightarrow$ gives formulas $\mathcal{I}(\Sigma_i \Rightarrow \Pi_i)$ derivable in $\mathrm{LK}(\mathcal{A})$ for some $i = 1, \ldots, m$

$\Rightarrow$ apply induction hypothesis $\vdash_{\mathrm{GK}(\mathcal{A})} \Sigma_i \Rightarrow \Pi_i$

$\Rightarrow$ apply $(sc_k), (MIX)$ and $(\Box_n)$ to get $\vdash_{\mathrm{GK}(\mathbb{R})} \Box\Gamma' \Rightarrow \Box\Delta'$

Main Theorem 2

Recall

$\mathfrak{KT}$ is $\mathfrak{K}$ plus $\Box\varphi \to \varphi$

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Recall $\mathfrak{KT}$ is $\mathfrak{K}$ plus $\Box\varphi \to \varphi$

Theorem 2 For any $\varphi \in \mathrm{Fm}$: $\vdash_{\mathfrak{KT}} \varphi \quad \Leftrightarrow \quad \models_{\mathrm{KT}(\mathcal{A})} \varphi$

Main Theorem 2

Recall $\mathfrak{KT}$ is $\mathfrak{K}$ plus $\Box\varphi \to \varphi$

Theorem 2 For any $\varphi \in \mathrm{Fm}$: $\quad \vdash_{\mathfrak{KT}} \varphi \quad \Leftrightarrow \quad \models_{\mathrm{KT}(\mathcal{A})} \varphi$

Proof same idea as for $\mathrm{K}(\mathcal{A})$

The two calculi

$\mathrm{LKT}(\mathcal{A})$ replace (ex) by

$$\frac{rjj}{rij}\ (ref)$$

The two calculi

$\mathrm{LKT}(\mathcal{A})$ replace (*ex*) by

$$\frac{rjj}{rij}\ (ref)$$

$\mathrm{GKT}(\mathcal{A})$ add

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}\ (\Box \Rightarrow)$$

Theorems

Theorem For any $\varphi \in \mathrm{Fm}$

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Theorem For any $\varphi \in \mathrm{Fm}$

$$\models_{\mathfrak{KT}} \mathcal{I}(\Gamma \Rightarrow \Delta) \qquad \Leftrightarrow \qquad \vdash_{\mathrm{GKT}(\mathcal{A})} \Gamma \Rightarrow \Delta$$

What's different to before?

before

$\Box$'s were removed simultaneously on the left and the right hand side of a sequent

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now $\Box$ can be removed just on left hand side by $(\Box \Rightarrow)$

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before $\Box$'s were removed simultaneously on the left and the right hand side of a sequent

now $\Box$ can be removed just on left hand side by $(\Box \Rightarrow)$

$\Rightarrow$ it is more complicated to apply induction hypothesis

Further Work

- Find axiomatizations for transitive (and symmetric) case

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- add lattice operators $\wedge$ and $\vee$

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- Find axiomatizations for transitive (and symmetric) case
- add lattice operators $\wedge$ and $\vee$
- Find complexitiy class of decidability for $\mathrm{K}(\mathcal{A})$ (is it EXPTIME-complete?)

References

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Axiom system $\mathrm{K}(\mathcal{A})$

axioms	(B)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
	(C)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
	(I)	$\varphi \to \varphi$
	(A)	$((\varphi \to \psi) \to \psi) \to \varphi$
	(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
	$(\mathrm{D_n})$	$\Box(n\varphi) \to n\Box\varphi \qquad (n \geq 2)$
rules	(mp)	$\{\varphi, \varphi \to \psi\}/\psi$
	(nec)	$\varphi/\Box\varphi$
	$(\mathrm{con_n})$	$n\varphi/\varphi \qquad (n \geq 2)$