# Projective WS5-Algebras

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# Outline

Why **WS5**-algebras?

Free **WS5**-algebra of Rank 1.

Projective Finitely Presented **WS5**-Algebras.

Primitive Quasivarieties of **WS5**-algebras.

Some Generalizations

# Logic **WS5** and it's Extentions

For modal intuitionistic logics, the logic **WS5** plays a role similar to the role played by the classical logic for superintuitionistic propositional logics. This similarity became even more apparent after in [Bezhanishvili, 2001]) Glivenko's Theorem had been extended to MIPC.

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Logic **WS5** an its extensions can also be viewed as a meta-logics for corresponding multiple-conclusion superintuitionistic logics. This correspondence preserves a lot of important properties (like decidability, finite axiomatizability, local tabularity, etc.)

# WS5-Algebras

The algebraic models for WS5: the Heyting algebras equipped with $\Box$ and the open elements form a Boolean algebra - the **WS5**-*algebras*, that is algebras $\mathbf{A} = (\mathsf{A}; \wedge, \vee, \rightarrow, \mathbf{1}, \mathbf{0}, \Box)$, where $(\mathsf{A}; \wedge, \vee, \rightarrow, \mathbf{1}, \mathbf{0})$ is a Heyting algebra and $\Box$ satisfies the following conditions

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\begin{array}{ll}
(M0) & \Box\mathbf{1} \approx \mathbf{1}; \\
(M1) & \Box x \rightarrow x \approx \mathbf{1}; \\
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The subdirectly irreducible (s.i.) **WS5**-algebras are exactly the **WS5**-algebras having two open elements: $\mathbf{0}$ and $\mathbf{1}$.

Free **WS5**-algebra of Rank 1

# Free Algebras of Rank 1 in Finitely Approximated Variety

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Every free algebra $\mathbf{F}_{\mathcal{V}}(n)$ is finitely approximated, that is, $\mathbf{F}_{\mathcal{V}}(n)$ is a subdirect product of finite s.i. algebras. Hence, in order to construct $\mathbf{F}_{\mathcal{V}}(1)$, it is enough

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- to construct the direct product of them
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For instance, if $\mathcal{H}$ is a variety of all Heyting algebras, in order to construct $\mathbf{F}_{\mathcal{H}}(1)$, one can take a direct product $\mathbf{P}$ all s.i. finite single-generated algebras $\mathbf{Z}_{2i+1}, i > 1$ and take a subalgebra $\mathbf{Z}$ generated by the element $(\mathbf{g}_3, \mathbf{g}_5, \dots)$, where $\mathbf{g}_{2i+1}$ is a generator of $\mathbf{Z}_{2i+1}$.

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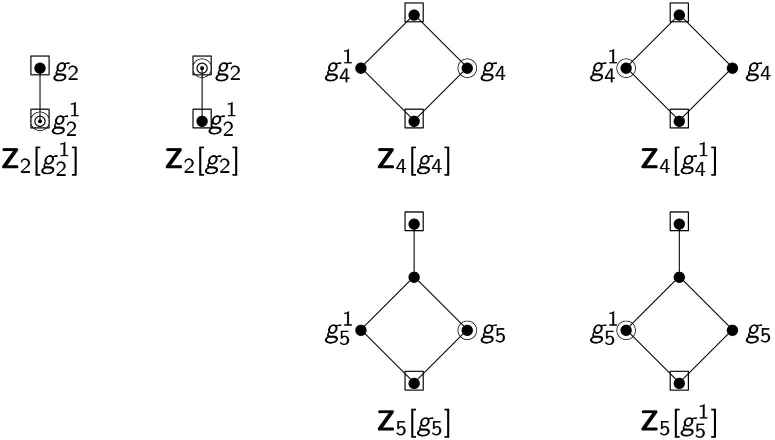


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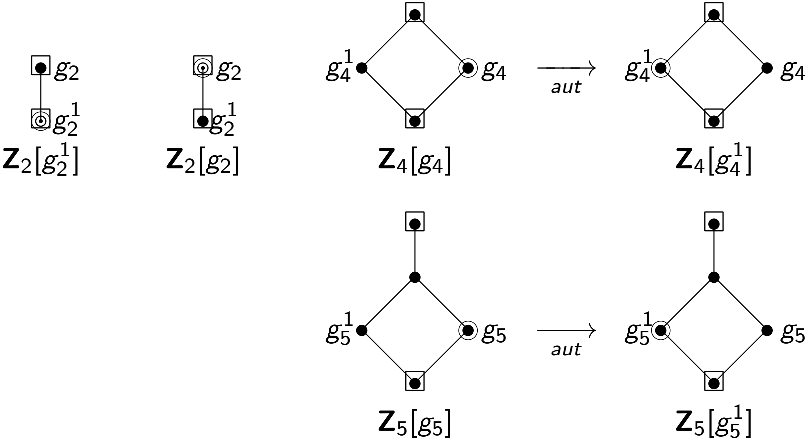


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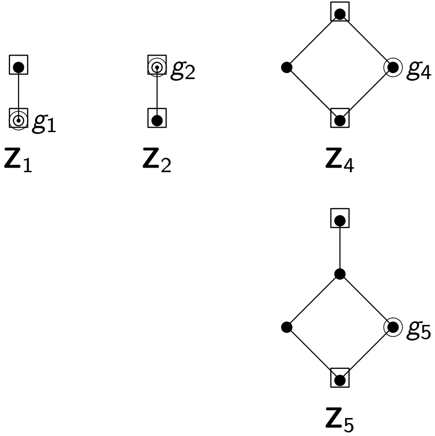


# Free Algebras of Rank 1

We have a set $\{\mathbf{Z}_k, k > 0\}$ of s.i. **WS5**-algebras generating $\mathcal{M}$, and each algebra $\mathbf{Z}_k$ is generated by $g_k$. Hence, the subalgebra $\mathbf{Z}$ of

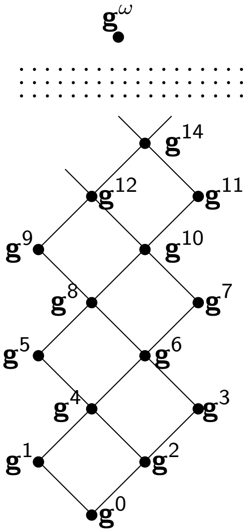
$$\mathbf{P} := \prod_{k>0} \mathbf{Z}_k$$

generated by element

$$\mathbf{g} = (g_1, g_2, \dots). \tag{1}$$

is isomorphic to $\mathbf{F}_{\mathcal{M}}(1)$.

# Degrees of Elements




$\mathbf{F}_{\mathcal{H}}(1)$

# Leveled Elements

## Definition

Let $k > 0$ and $m \in \{0, 1, \ldots, \omega\}$. An element $\mathbf{a} \in \mathbf{P}$ is called $(k, m)$-*leveled*, if for all $i \geq k$,

$$\pi_i(\mathbf{a}) = \mathbf{g}_i^m.$$

An element $\mathbf{a} \in \mathbf{P}$ is $(k, m)$-leveled if, starting from $k$-th component, each component of $\mathbf{a}$ is equal to the same degree of the respective generator, that is, $\mathbf{a}$ is of form

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**Definition**

An element $\mathbf{a} \in \mathbf{P}$ is *leveled*, if it is $(k, m)$-leveled for some $k > 0$ and $m \in \{0, 1, \dots, \omega\}$.

For instance, if $\mathbf{a}$ is a *binary element*, that is each component of $\mathbf{a}$ is $\mathbf{0}$ or $\mathbf{1}$, then $\mathbf{a}$ is leveled if and only if it contains either a finite number of $\mathbf{0}$-components, or a finite number of $\mathbf{1}$-components.

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An element $\mathbf{a} \in \mathbf{Z}$ is open if and only if $\mathbf{a}$ is binary, hence $\mathbf{a}$ is open if and only if it contains either a finite number of $\mathbf{0}$-components, or a finite number of $\mathbf{1}$-components.

# Some Properties of $\mathbf{F}_{\mathcal{M}}(1)$

Corollary (Comp. [Grigolia, 1995, Theorem 5.2])

*Algebra* $\mathbf{F}_{\mathcal{M}}(1)$ *is atomic and has infinitely many atoms.*

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**Corollary**

*Algebra $\mathbf{F}_{\mathcal{M}}(1)$ has infinite ascending and descending chains of open elements.*

Projective Finitely Presented **WS5**-Algebras

# Total Non-Projectivity

An algebra $\mathbf{A}$ is *projective* in a variety $\mathcal{V}$ if for any algebra $\mathbf{B} \in \mathcal{V}$ and any homomorphism $\varphi : \mathbf{B} \longrightarrow \mathbf{A}$ there is an embedding $\psi : \mathbf{A} \longrightarrow \mathbf{B}$ such that $\psi \circ \phi = id_{\mathbf{A}}$, where $id_{\mathbf{A}}$ is an identity map.

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**Corollary**

*Every s.i.* **WS5***-algebra* $\mathbf{A}$ *distinct from* $\mathbf{2}$ *is totally non-projective.*

# Projective Finitely Presented Algebras

### Theorem

*Let $\mathcal{V} \subseteq \mathcal{M}$ be a variety of* **WS5***-algebras and* **A** *be a nontrivial finitely presented in $\mathcal{V}$ algebra. Then the following is equivalent*

(a) **A** *is projective in $\mathcal{V}$;*

(b) **A** *does not contain an element* **a** *such that*

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## Remark

From [Quackenbush, 1971, Theorem 5.2] it follows that if $\mathcal{V} \subseteq \mathcal{M}$ is a variety generated by a quasi-primal algebra, then a finite algebra $\mathbf{A} \in \mathcal{V}$ is projective if and only if **2** is its direct factor.

**Corollary**

*In any subvariety $\mathcal{V} \subseteq \mathcal{M}$ every finitely presented subalgebra of $\mathbf{F}_{\mathcal{V}}(\omega)$ is projective. In particular, every finite subalgebra of $\mathbf{F}_{\mathcal{V}}(\omega)$ is projective*

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**Corollary**

*Let $\mathcal{V}$ be a variety of* **WS5***-algebras and $\mathbf{A} \in \mathcal{V}$ be a finitely presented algebra given by relation $t = r$. Then $\mathbf{A}$ is projective if and only if $t = r$ is satisfiable in $\mathbf{2}$.*

Primitive Quasivarieties of **WS5**-algebras

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A quasivariety $\mathcal{Q}$ is *primitive* or *deductive* if every its subquasivariety is a relative variety, that is, if every subquasivariety of $\mathcal{Q}$ can be defined relative to $\mathcal{Q}$ by a set of identities (see [Gorbunov, 1998]). The above Theorem gives us a way to characterize all primitive quasivarieties of **WS5**-algebras.

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An algebra $\mathbf{A}$ does not have element $\mathbf{a}$ such that $\Box\mathbf{a} = \Box\neg\mathbf{a}$ if and only if quasi-identity

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**Corollary**

*An algebra* **A** *finitely presented in a variety* $\mathcal{V} \subseteq \mathcal{M}$ *is projective if and only if* $\mathbf{A} \models \rho$.

# Primitive Quasivarieties of **WS5**-Algebras

Any primitive quasivariety $\mathcal{Q}$ is generated by free algebra $\mathbf{F}_{\mathcal{V}}(\omega)$ of variety $\mathcal{V}$ generated by $\mathcal{Q}$.

**Theorem**

*Let $\mathcal{V}$ be a variety of* **WS5***-algebras and $\mathcal{Q}$ be a quasivariety generated by $\mathbf{F}_{\mathcal{V}}(\omega)$. Then $\rho$ defines $\mathcal{Q}$ relative to $\mathcal{V}$.*

**Corollary**

*A quasivariety of* **WS5***-algebras is primitive if and only if it admits quasi-identity $\rho$.*

# Primitive Quasivarieties of **WS5**-Algebras

### Remark

Quasi-identity $\rho$ is an algebraic version of a passive inference rule

$$\Diamond p \wedge \Diamond \neg p / \bot$$

introduced for normal modal logics extending S4.3 by Rybakov (see [Rybakov, 1984]). In algebraic terms, Theorem 5 from [Rybakov, 1984] gives a characterization of primitive quasivarieties of S4.3-algebras. The passive inference rules in logics from **Ext**S4.3 are extensively studied in [Dzik & Wojtylak, 2016].

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Let $\mathcal{V}$ be a variety of algebras of an arbitrary finite similarity type. A nontrivial algebra $\mathbf{A} \in \mathcal{V}$ is *minimal* if $\mathbf{A}$ does not contain proper subalgebras. $\mathcal{V}_{min}$ denotes a set of all minimal algebras from $\mathcal{V}$.

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### Corollary

*Every projective algebras from* $\mathcal{V}$ *is mh-full.*

For instance, every finite projective Łukasiewicz algebra has the two-element Łukasiewicz algebra as a homomorphic image (comp. [Di Nola et al., 2008]).

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**Theorem**

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**Corollary**

*Every finitely-presented subalgebra of $\mathbf{F}_{\mathcal{V}}(\omega)$ is projective. In particular, every finite subalgebra of $\mathbf{F}_{\mathcal{V}}(\omega)$ is projective.*

# Some Generalizations

The above criterion holds in every ms-full discriminator variety, for in discriminator varieties every compact congruence is principal and each principal congruence is a factor congruence (see e.g. [Andréka et al., 1991]).

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If $\mathbf{F}_\omega(\mathcal{V})$ has a finite nontrivial subalgebra and all nontrivial algebras from $\mathcal{V}$ do not have trivial subalgebras, then $\mathcal{V}$ is ms-full. For instance, the criterion holds in every double-pointed discriminator variety $\mathcal{V}$ as long as $\mathbf{F}_\omega(\mathcal{V})$ has finite subalgebras.

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For instance, the criterion holds in the discriminator varieties of

- Heyting algebras with pseudocomplementation;
- double Heyting algebras;
- Heyting algebras with involution (simmetrical Heyting algebras);
- **WS5**-algebras with compatible operations if $\{\mathbf{0}, \mathbf{1}\}$ forms a subalgebra of a free algebra.

## Corollary

*(comp. [Dzik & Wojtylak, 2016, Corollary 3.1] for* S4.3*)*
*Suppose $\mathcal{V}$ is an ms-full discriminator variety and* $\mathbf{A}$ *is a finitely presented algebra defined in $\mathcal{V}$ by relation $t = r$. Then the following is equivalent*

(a) $\mathbf{A}$ *is projective in $\mathcal{V}$;*

(b) *$t = r$ is satisfiable in every minimal algebra from $\mathcal{V}$;*

(c) *$t = r$ is unifiable in $\mathbf{F}_{\mathcal{V}}(\omega)$.*

# Thank You

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