Sasaki projections and related operations

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What is quantum logic?

Crucial example:

- The lattice of closed subspaces of a separable Hilbert space $\mathcal{H}$
- $x \wedge y = x \cap y$
- $x' =$ the closure of $\{\mathbf{u} \mid \mathbf{u} \perp \mathbf{v} \text{ for all } \mathbf{v} \in x\}$
- $x \vee y = (x' \wedge y')'$

Orthomodular lattice

More generally [Birkhoff, von Neumann 1936]:

Definition

An *orthomodular lattice* is a bounded lattice with an *orthocomplementation* $'$ satisfying

- $x \leqslant y \Rightarrow y' \leqslant x'$
- $x'' = x$
- x' is the lattice-theoretical complement of x:

$$\begin{aligned} x \wedge x' &= \mathbf{0} \\ x \vee x' &= \mathbf{1} \end{aligned}$$

- $x \leqslant y \Rightarrow y = x \vee (x' \wedge y)$ (*orthomodular law*)

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- Whether $x = y$, $\quad x \le y$, $\quad x = y'$, $\quad x \perp y$ (i.e., $x \le y'$).
- In all these (and some other) cases, x, y generate a finite Boolean subalgebra; we say that x, y *commute*; in symbols, $x \, \mathrm{C} \, y$.

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- We can describe at least the *orthogonal projection* of y to x,

$$x \wedge (x' \vee y) = \phi_x(y) = x * y$$

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- $x \,\mathrm{C}\, y \implies \phi_x(y) = x \wedge y$

Sasaki (binary) operation

The Sasaki operation is neither commutative nor associative, it satisfies

idempotence	$x * x$	$=$	x		
neutral element	$\mathbf{1} * x$	$=$	$x * \mathbf{1}$	$=$	x
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The Sasaki operation and its dual, *Sasaki hook*, may be better candidates for the conjunction and disjunction of a quantum logic than the meet and join [Pykacz 2015].

Weaker forms of associativity

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Theorem (Alternative algebra)

An OML with the Sasaki operation forms an alternative algebra, i.e.,

$$\begin{array}{rcll} x * (x * y) & = & (x * x) * y & \textit{(left identity)} \\ (y * x) * x & = & y * (x * x) & \textit{(right identity)} \\ x * (y * x) & = & (x * y) * x & \textit{(flexible identity)} \end{array}$$

Theorem (Moufang–like identities)

$$\begin{array}{rcl} (x * y * x) * z & = & (x * y) * (x * z) \\ \big(z * (x * y)\big) * x & = & z * (x * y * x) \\ \big((x * y) * z\big) * x & = & (x * y) * (z * x) \end{array}$$

Properties of Sasaki projection

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- The *dual* of a monotonic mapping θ is

$$\overline{\theta}(y) = (\theta(y'))'\,.$$

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- $\phi_p \phi_q = \phi_q \phi_p = \phi_p \iff p \leqslant q$

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- Each $\xi = \phi_{x_n} \cdots \phi_{x_2}\phi_{x_1} \in S(L)$ has a unique *adjoint*

$$\xi^*(y) = \min\{z \in L \mid \overline{\xi}(z) \geq y\},$$

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Problem: Prove this without the advanced methods of Baer *-semigroups.

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Here the meet $\wedge$ cannot be used instead of Sasaki operation $*$.

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Question [Chevalier, Pulmannová 1992]: Are $\xi(\mathbf{1}), \xi^*(\mathbf{1})$ (strongly) perspective? (Here $\xi = \phi_{x_n} \cdots \phi_{x_2}\phi_{x_1}, \xi^* = \phi_{x_1}\phi_{x_2} \cdots \phi_{x_n}$.)

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- The potential of using Sasaki projections in the algebraic foundations of orthomodular lattices is still not sufficiently exhausted.

Thank you for your attention!