Topological spaces of monadic MV-algebras

R. Grigolia (Tbilisi), A. Di Nola and G. Lenzi (Salerno)

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Summary

We generalize Belluce β functor from MV-algebras to distributive lattices

we obtain a functor γ from monadic MV-algebras to Q-distributive lattices (hence to Q-spaces via Cignoli duality)

and we introduce a subcategory of Q-spaces related to γ (monadic Q-spaces).

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Motivation: many valued logic

Idea: truth values change from $\{0, 1\}$ to $[0, 1]$

$(\{0, 1\}, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra

what structure can we put on $[0, 1]$?

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The MV-algebra $[0, 1]$

As negation we have $\neg x = 1 - x$ (the Liar tells half the truth)
As disjunction we have $x \oplus y = min(1, x + y)$ (not idempotent, cfr. Ulam games with lies)
Conjunction $x \odot y$ is defined via De Morgan duality.

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MV-algebras (introduced by Chang) are the algebraic counterpart of propositional infinite valued Łukasiewicz logic.

Now full first order Łukasiewicz logic is not axiomatizable (Scarpellini)

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They are structures $A = (X, \oplus, \neg, 0, 1)$, where

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- $\neg\neg x = x$;
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The Łukasiewicz product

The derived operation

$$x \odot y = \neg(\neg x \oplus \neg y)$$

is the De Morgan dual of the sum $\oplus$.

The partial order of an MV-algebra

We let $x \leq y$ if $y = x \oplus z$ for some z.

This is a lattice with
$$x \vee y = \neg(\neg x \oplus y) \oplus y$$
and
$$x \wedge y = \neg(\neg x \vee \neg y).$$
A linearly ordered MV-algebra is called an MV-chain.

Ideals

An ideal of an MV-algebra A is a set $I \subseteq A$ such that

- $x, y \in I$ implies $x \oplus y \in I$
- $x \in I$ and $y \leq x$ imply $y \in I$.

An ideal I is prime if $I \neq A$ and whenever $x \wedge y \in I$ we have $x \in I$ or $y \in I$.

Ideals and prime ideals can be also defined in lattices.

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Filters

A filter of an MV-algebra A is a set $F \subseteq A$ such that

- $x, y \in F$ implies $x \odot y \in F$
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A filter F is prime if $F \neq A$ and whenever $x \vee y \in F$ we have $x \in F$ or $y \in F$.
Filters and prime filters can be also defined in lattices.

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The prime spectrum of a lattice

Given a lattice L, $Spec(L)$ is the set of all prime filters of L whose topology (Zariski topology) is generated by the opens $U_a = \{F \in Spec(L) | a \in F\}$.

The importance of $[0, 1]$

Theorem (Di Nola embedding)

Every MV algebra embeds in a power of an ultrapower of $[0, 1]$.

Corollary

$[0, 1]$ generates the variety of MV algebras.

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Finite MV-algebras

The finite chains are $S_n = \{0, 1/n, 2/n, \ldots, n-1/n, 1\}$.
Every finite MV-algebra is a finite product of chains.

The spectrum problem for MV-algebras

We do not have a good topological characterization of spectra of MV-algebras
(we have it as ordered sets thanks to Cignoli-Torrens, and we have it for countable MV-algebras thanks to Wehrung).
One of the tools devised for this problem is Belluce functor, which replaces the MV-algebras with “simpler” objects.

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The Belluce β functor

Given MV-algebra A we define the equivalence $x \equiv y$ if x and y belong to the same prime ideals.

Define $\beta(A) = A/\equiv$, which has a natural structure of a lattice.

Moreover, the prime spectra of A and $\beta(A)$ are homeomorphic.

β can be extended to a functor from MV-algebras to bounded distributive lattices by letting $\beta(f)(\beta(x)) = \beta(f(x))$.

If we consider filters rather than ideals, we obtain a dual functor β^*.

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The category *MMV* of Monadic MV-algebras

Monadic MV-algebras are structures $(A, \exists)$, where A is an MV-algebra, $\exists : A \to A$ and

- $x \leq \exists x$
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- $\exists \neg(\exists x) = \neg \exists x$
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Monadic Boolean algebras are dual to Boolean spaces with Boolean equivalence relations.
We do not have, instead, a duality for the full category of monadic MV-algebras (not even of MV-algebras).

Monadic Boolean algebras (Halmos)

Recall that Boolean algebras are idempotent MV-algebras ($x \oplus x = x$)
In the same vein, Monadic Boolean algebras are structures $(A, \exists)$, where A is a Boolean algebra, $\exists : A \to A$ and

- $x \leq \exists x$
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Structure of monadic MV-algebras

If $(A, \exists)$ is a monadic MV-algebra then $A_0 = \{x | \exists x = x\}$ is an MV-subalgebra of A which is m-relatively complete, that is,

- for every $a \in A$ the infimum $inf\{b \in A_0 | b \geq a\}$ exists in A_0
- if $a \in A, x \in A_0, x \geq a \odot a$ then there is $v \in A_0$ with $v \geq a$ and $x \geq v \odot v$
- if $a \in A, x \in A_0, x \geq a \oplus a$ then there is $v \in A_0$ with $v \geq a$ and $x \geq v \oplus v$.

Conversely, every mrc-subalgebra A_0 of A gives a quantifier by letting $\exists a = inf\{b \in A_0 | b \geq a\}$.

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Monadic ideals

A monadic ideal of $(A, \exists)$ is an MV-algebra ideal closed under $\exists$.
There is an isomorphism between:

- the lattice of monadic ideals of $(A, \exists)$;
- the lattice of congruences of $(A, \exists)$;
- the lattice of ideals of $\exists A$.

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Birkhoff subdirect representation

Theorem

(Rutledge) Every monadic MV-algebra $(A, \exists)$ is a subdirect product of monadic MV-algebras $(A_i, \exists_i)$ where $\exists_i A_i$ is totally ordered.

The totally ordered case is trivial

Lemma

If A_0 is an m-relatively complete totally ordered MV-subalgebra of an MV-algebra A, then A_0 is a maximal totally ordered subalgebra of A.

Corollary

If $(A, \exists)$ is totally ordered then $A = \exists A$.

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An example of monadic MV-algebra

A diagonal construction:
$A = [0,1]^n$
$\exists(x_1, \ldots, x_n) = (m, m, \ldots, m)$
where $m = max\{x_1, \ldots, x_n\}$.

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The finite case

Theorem

If $(A, \exists)$ is a finite monadic MV-algebra with totally ordered $\exists A$, then $A = (\exists A)^n$ and $\exists(x_1, \ldots, x_n) = (m, m, \ldots, m)$, where $m = max\{x_1, \ldots, x_n\}$.

More generally, if $A = S_{n_1} \times \ldots \times S_{n_k}$ is finite, then monadic structures can be found by considering homogeneous partitions of $\{1, \ldots, k\}$, that is partitions where two equivalent indices correspond to equal chains. On each block of the partition, one can perform the diagonal construction.

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Dualities

In duality theory, “abstract” algebraic objects are put in correspondence with “concrete” geometric or topological objects. The theory of lattices gives a huge amount of examples. Here we will only recall some of them.

Calculate! (Leibniz)
Topologize! (Stone)

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Calculate! (Leibniz)
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Priestley spaces

Priestley discovered a duality between the category of bounded distributive lattices and the category of Priestley spaces, extending Stone duality for Boolean algebras.

A Priestley space is a structure (X, R), where X is a compact topological space and R is an order relation on X such that, for all $x, y \in X$, either xRy or there is a clopen up-set V with $x \in V$ and $y \notin V$.

We denote by $P(X)$ the set of clopen up-sets of X.

A morphism of Priestley spaces is a continuous, order preserving map.

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Priestley duality

The dual of L is $(Spec(L), \subseteq)$ where $Spec(L)$ is the prime spectrum of L equipped with the patch topology (the one generated by $\{P|a \in P\}$ and $\{P|a \notin P\}$ for $a \in L$).
The dual of (X, R) is $P(X)$.
In both senses, the duality on morphisms is given by the inverse image.
What if quantifiers are added?

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The category QD of Q-distributive lattices

Intuitively, Q-distributive lattices are negation-free monadic Boolean algebras.

Q-distributive lattices (Cignoli) are structures $(L, \exists)$ where L is a bounded distributive lattice, $\exists : L \to L$ and

- $\exists 0 = 0$
- $a \wedge \exists a = a$
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Cignoli found a dual category to Q-distributive lattices: Q-spaces.

A Q-space (Cignoli) is a structure (X, R, E) where (X, R) is a Priestley space and E is an equivalence on X such that

- For every $U \in P(X)$ we have $E(U) \in P(X)$
- The equivalence classes of E are closed in X.

A morphism of Q spaces (X, R, E) and (Y, S, F) is a map $f : X \to Y$ which is continuous, order preserving and such that $E(f^{-1}(V)) = f^{-1}(F(V))$ for every $V \in P(Y)$.

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The functor Q^* from QD to QD^*

We define $Q^*(L, \exists) = (Spec(L), \subseteq, E(L, \exists))$ where:

- $Spec(L)$ is the prime spectrum of L, i.e. the set of prime filters of L
- the topology on $Spec(L)$ is the patch topology
- $\subseteq$ is the inclusion relation in $Spec(L)$ (so $(Spec(L), \subseteq)$ is a Priestley space)
- $E(L, \exists) = \{(F, G) \in Spec(L)^2 | F \cap \exists L = G \cap \exists L\}$.

Given a morphism of Q-distributive lattices $h : A \to B$, we define $Q^*(h)$ by
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The pair (Q, Q^*) is a duality between QD and QD^* (Cignoli).

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The functor γ from *MMV* to *QD*

γ is an extension of Belluce β functor.

In fact, $\gamma(A, \exists) = (A/\equiv', \exists')$ where $x \equiv' y$ if x and y belong to the same prime ideals of A,
and $\exists x$ and $\exists y$ belong to the same prime ideals;
moreover $\exists'[a] = [\exists a]$ where $[a]$ is the equivalence class of a modulo $\equiv'$.

γ becomes a functor from MMV-algebras to Q-distributive lattices by $\gamma(f)(\gamma(x)) = \gamma(f(x))$.
Like for β, the prime spectra of $(A, \exists)$ and $\gamma(A, \exists)$ are homeomorphic.

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The codomain of γ

From the theory of prime spectra of MV-algebras it follows that the co-domain of the functor γ is given by the dual completely normal distributive lattices, that is (Wehrung)

for every a, b there are x, y such that $a \geq b \wedge x$, $b \geq a \wedge y$ and $x \vee y = 1$.

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Summing up

By composing γ with Q^*, we obtain a functor from monadic MV-algebras to Q-spaces,
actually monadic Q-spaces which we define right now.

The category of monadic Q-spaces

A monadic Q-space is a Q-space (X, R, E) such that

- $R(x)$ is a chain for every x
- $RE(x) = ER(x)$
- $R^{-1}E(x) = ER^{-1}(x)$
- $R(x) \cap E(x) = R^{-1}(x) \cap E(x) = \{x\}$.

A morphism of monadic Q-spaces is a strongly isotone mapping of Q-spaces.
Recall that a monotonic map $f : X \to Y$ between spaces (X, R, E) and (Y, S, F) is strongly isotone if
$R(f(x)) = f(S(x))$ for every $x \in X$.

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- $R(x) \cap E(x) = R^{-1}(x) \cap E(x) = \{x\}$.

A morphism of monadic Q-spaces is a strongly isotone mapping of Q-spaces.
Recall that a monotonic map $f : X \to Y$ between spaces (X, R, E) and (Y, S, F) is strongly isotone if
$R(f(x)) = f(S(x))$ for every $x \in X$.

If you want to know more...

Antonio Di Nola: adinola@unisa.it

Revaz Grigolia: revaz.grigolia@tsu.ge

Giacomo Lenzi: gilenzi@unisa.it

Thank you!