A simple restricted Priestley duality for distributive lattices with an order-inverting operation

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Some history and some excuses

There are many predecessors doing "something similar but in a different direction". There is at least one predecessor doing more:

- J. Farley, *Priestley Duality for Order-Preserving Maps into Distributive Lattices*, Order **13**, 65–98, 1996.

Farley's work uses fairly advanced topology.

- Our work was done independently, out of laziness and negligence.
- It does not require advanced techniques, beyond Priestley duality and basic categorical notions.
- It is an example of a restricted Priestley duality as defined in B.A. Davey, A. Gair, *Restricted Priestley Dualities and Discriminator Varieties*
- It can be used to investigate algebraically "the logic of minimal negation" (and the lattice of subvarieties of the corresponding variety of algebras).

BDLs with order-inverting operation

A bounded distributive lattice with order-inverting operation (or BDL with negation), is an algebra $\mathrm{A} = (A; \wedge, \vee, \neg, 0, 1)$, such that

- $(A; \wedge, \vee, 0, 1)$ is a bounded distributive lattice, and
- $\neg$ is an order-inverting operation.

Let BDLN be the class of all such algebras.

Lemma

The class BDLN *is precisely the class of bounded distributive lattices with a unary operation* $\neg$ *satisfying the following weak De Morgan laws*

$$\neg x \vee \neg y \leq \neg(x \wedge y),$$
$$\neg(x \vee y) \leq \neg x \wedge \neg y.$$

Thus, $\mathbb{BDLN}$ *is a variety.*

Logic of minimal negation

A sequent is a pair of multisets of terms. As usual, we begin by specifying initial sequents:

$$\vdash 1 \qquad \alpha \vdash \alpha \qquad 0 \vdash$$

As structural rules, we take left and right weakening:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \alpha, \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta}$$

left and right contraction:

$$\frac{\Gamma \vdash \alpha, \alpha, \Delta}{\Gamma \vdash \alpha, \Delta} \qquad \frac{\Gamma, \alpha, \alpha \vdash \Delta}{\Gamma, \alpha \vdash \Delta}$$

and unrestricted cut:

$$\frac{\Gamma \vdash \alpha, \Delta \quad \Sigma, \alpha \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}$$

Logic of minimal negation

Next, the introduction rules for $\wedge$ and $\vee$:

$$\frac{\Gamma, \alpha \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta} \qquad \frac{\Gamma, \beta \vdash \Delta}{\Gamma, \alpha \wedge \beta \vdash \Delta} \qquad \frac{\Gamma \vdash \alpha, \Delta \quad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta}$$

$$\frac{\Gamma \vdash \alpha, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta} \qquad \frac{\Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \vee \beta, \Delta} \qquad \frac{\Gamma, \alpha \vdash \Delta \quad \Gamma, \beta \vdash \Delta}{\Gamma, \alpha \vee \beta \vdash \Delta}$$

Up to here, everything is classical. Now, for negation we assume only the minimal

$$\frac{\alpha \vdash \beta}{\neg\beta \vdash \neg\alpha}$$

instead of the classical

$$\frac{\Gamma, \alpha \vdash \Delta}{\Gamma, \vdash \neg\alpha, \Delta} \qquad \frac{\Gamma \vdash \beta, \Delta}{\Gamma, \neg\beta \vdash \Delta}$$

The logic and the variety

Curios

Let L be the logic defined above.

1. L is not algebraizable in the sense of Blok-Pigozzi.
2. L is not order-algebraizable in the sense of Raftery.
3. L is algebraizable as a sequent system, in the sense of Rebagliato-Verdú and Blok-Jónsson. Thus, BDLN is a natural semantics of L.
4. BDLN is not point-regular.
5. BDLN has the finite embeddability property.
6. The lattice reduct of the free zero-generated algebra in BDLN is a chain has order type $\omega + \omega^*$.
7. Cut elimination holds in L.

The dual category: objects

Definition

The objects are pairs $\big(P, \mathcal{N}\colon P \to \mathcal{O}(\mathrm{ClopUp}(P))\big)$, where

1. P is a Priestley space.
2. $\mathrm{ClopUp}(P)$ is the set of clopen up-sets of P.
3. $\mathcal{O}(\mathrm{ClopUp}(P))$ is the set of downsets of $\mathrm{ClopUp}(P)$.
4. $\mathcal{N}\colon P \to \mathcal{O}(\mathrm{ClopUp}(P))$ is an order-preserving map, such that for every $X \in \mathrm{ClopUp}(P)$, the set $\{p \in P \colon X \in \mathcal{N}(p)\}$ is clopen.

- $\{p \in P \colon X \in \mathcal{N}(p)\}$ will be $\neg X$.
- If P is finite, then $\mathrm{ClopUp}(P)$ is just the set of up-sets of P, and (4) is satisfied by any order-preserving map.

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- If P is finite, then $\mathrm{ClopUp}(P)$ is just the set of up-sets of P, and (4) is satisfied by any order-preserving map.
- Example: the simplest that can be...

The dual category: preparing for morphisms

- Any order-preserving map $h\colon P \to Q$ between ordered sets P and Q can be naturally lifted to a map $h^{-1}\colon \mathcal{P}(Q) \to \mathcal{P}(P)$ taking each $X \in \mathcal{P}(Q)$ to $h^{-1}(X) \in \mathcal{P}(P)$.
- h^{-1} maps up-sets to up-sets and downsets to downsets.
- The lifting can be iterated. E.g., $(h^{-1})^{-1}\colon \mathcal{P}(\mathcal{P}(P)) \to \mathcal{P}(\mathcal{P}(Q))$. We will write $\overline{h}$ for this double lifting.
- $\overline{h}$ maps up-sets to up-sets and downsets to downsets.
- Let $(P, \mathcal{N}^P)$ and $(Q, \mathcal{N}^Q)$ be objects, and let $h\colon P \to Q$ be a continuous map. Since h is continuous, the map $h^{-1}\colon \mathrm{ClopUp}(Q) \to \mathrm{ClopUp}(P)$ is well defined.
- Thus, $\overline{h}$ is also well defined as a map from $\mathcal{O}(\mathrm{ClopUp}(P))$ to $\mathcal{O}(\mathrm{ClopUp}(Q))$.

The dual category: morphisms

- Let $h\colon P \to Q$ be a continuous order-preserving map. Then, for any $W \in \mathcal{O}(\mathrm{ClopUp}(P))$, we have $\overline{h}(W) = \{U \in \mathrm{ClopUp}(Q) \colon h^{-1}(U) \in W\}$.

Definition

A morphism from $(P, \mathcal{N}^P)$ to $(Q, \mathcal{N}^Q)$ is a continuous order-preserving map $h\colon P \to Q$ such that the diagram below commutes.

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "\mathcal{N}^P"'] & Q \arrow[d, "\mathcal{N}^Q"] \\ \mathcal{O}(\mathrm{ClopUp}(P)) \arrow[r, "\overline{h}"] & \mathcal{O}(\mathrm{ClopUp}(Q)) \end{tikzcd}$$

Dual equivalence

Theorem

The categories BDLN (*with homomorphisms*) *and* $\mathbb{OTNS}$ *are dually equivalent.*

Define $E\colon \mathbb{OTNS} \to \mathsf{BDLN}$ as follows:

- For an object $\mathcal{P} \in \mathbb{OTNS}$, we put

$$E(\mathcal{P}) = \big(\mathrm{ClopUp}(P), \cup, \cap, \neg, \emptyset, P\big)$$

  where for every $X \in \mathrm{ClopUp}(P)$ we have

$$\neg X = \{p \in P \colon X \in \mathcal{N}(p)\}.$$

- For a morphism $h \in \mathrm{Hom}(\mathcal{P}, \mathcal{Q})$, we put

$$E(h)(U) = h^{-1}(U)$$

  for every $U \in \mathrm{ClopUp}(P)$.

Dual equivalence

Define $D\colon \mathsf{BDLN} \to \mathbb{OTNS}$, as follows:

- For an algebra $\mathsf{A} \in \mathsf{BDLN}$, we first take the usual Priestley topology on the set $\mathcal{F}_p(\mathsf{A})$ of all prime filters of A, and then, we put

$$D(\mathsf{A}) = \big(\mathcal{F}_p(\mathsf{A}),\ \mathcal{N}_{\mathsf{A}}\colon \mathcal{F}_p(\mathsf{A}) \to \mathcal{O}(\mathrm{ClopUp}(\mathcal{F}_p(\mathsf{A})))\big)$$

where for every $F \in \mathcal{F}_p(\mathsf{A})$ we have

$$\mathcal{N}_{\mathsf{A}}(F) = \big\{\{H \in \mathcal{F}_p(\mathsf{A})\colon a \in H\}\colon \neg a \in F\big\}.$$

- For a homomorphism $f \in \mathrm{Hom}(\mathsf{A}, \mathsf{B})$, we put

$$D(f) = f^{-1}$$

where $D(f)(G) = f^{-1}(G)$ for every $G \in \mathcal{F}_p(\mathsf{B})$.

Frame conditions

- Some examples of conditions on the algebras and corresponding conditions on dual spaces. Such things are known as frame conditions in dualities for BAOs.

	Algebra	Dual space
1	$\neg 1 = 0$	$\forall p \in P \colon P \notin \mathcal{N}(p)$
2	$\neg 0 = 1$	$\forall p \in P \colon P \notin \mathcal{N}(p)$
3	$\neg x$ is the pseudo-complement of x	$X \in \mathcal{N}(p)$ iff $\uparrow p \cap X = \emptyset$
4	$\neg$ is a dual endomorphism	$\forall p \in P \colon \mathcal{N}(p) \in im(P)$

where $im(P)$ is the image of P under the natural order-embedding of P into $\mathcal{O}(\mathcal{U}(P))$.

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where $im(P)$ is the image of P under the natural order-embedding of P into $\mathcal{O}(\mathcal{U}(P))$.

- The third condition corresponds to an intuitionistic negation, the fourth to a de Morgan negation (the algebras are known as Ockham lattices).

Lattice of subvarieties

- Level 1. There are 3 atoms: generated by the 3 algebras based on the 2-element chain.
- Level 2. Algebras based on the 3-element chain generate 5 more join-irreducible varieties (there are 3 more: varietal joins of the atoms).
- Level 3. Too messy to do by hand, perhaps. Conjecture: infinite.