A simple restricted Priestley duality for distributive lattices with an order-inverting operation

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Some history and some excuses

There are many predecessors doing “something similar but in a different direction”. There is at least one predecessor doing more:


Farley’s work uses fairly advanced topology.

- Our work was done independently, out of laziness and negligence.
- It does not require advanced techniques, beyond Priestley duality and basic categorical notions.
- It is an example of a restricted Priestley duality as defined in B.A. Davey, A. Gair, *Restricted Priestley Dualities and Discriminator Varieties*
- It can be used to investigate algebraically “the logic of minimal negation” (and the lattice of subvarieties of the corresponding variety of algebras).
BDLs with order-inverting operation

A bounded distributive lattice with order-inverting operation (or BDL with negation), is an algebra $A = (A; \land, \lor, \neg, 0, 1)$, such that

- $(A; \land, \lor, 0, 1)$ is a bounded distributive lattice, and
- $\neg$ is an order-inverting operation.

Let BDLN be the class of all such algebras.

**Lemma**

*The class BDLN is precisely the class of bounded distributive lattices with a unary operation $\neg$ satisfying the following weak De Morgan laws*

\[ \neg x \lor \neg y \leq \neg(x \land y), \]
\[ \neg(x \lor y) \leq \neg x \land \neg y. \]

*Thus, BDLN is a variety.*
Logic of minimal negation

A sequent is a pair of multisets of terms. As usual, we begin by specifying initial sequents:

\[ \vdash 1 \quad \alpha \vdash \alpha \quad 0 \vdash \]

As structural rules, we take left and right weakening:

\[ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \alpha, \Delta} \quad \frac{\Gamma \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \]

left and right contraction:

\[ \frac{\Gamma \vdash \alpha, \alpha, \Delta}{\Gamma \vdash \alpha, \Delta} \quad \frac{\Gamma, \alpha, \alpha \vdash \Delta}{\Gamma, \alpha \vdash \Delta} \]

and unrestricted cut:

\[ \frac{\Gamma \vdash \alpha, \Delta \quad \Sigma, \alpha \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi} \]
Logic of minimal negation

Next, the **introduction rules** for \( \land \) and \( \lor \):

\[
\begin{align*}
\Gamma, \alpha & \vdash \Delta \\
\Gamma, \alpha \land \beta & \vdash \Delta \\
\end{align*}
\]

\[
\begin{align*}
\Gamma, \beta & \vdash \Delta \\
\Gamma, \alpha \land \beta & \vdash \Delta \\
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash \alpha, \Delta \\
\Gamma & \vdash \alpha \lor \beta, \Delta \\
\end{align*}
\]

\[
\begin{align*}
\Gamma & \vdash \beta, \Delta \\
\Gamma & \vdash \alpha \lor \beta, \Delta \\
\end{align*}
\]

Up to here, everything is classical. Now, for negation we assume only the minimal

\[
\begin{align*}
\alpha & \vdash \beta \\
\neg \beta & \vdash \neg \alpha \\
\end{align*}
\]

instead of the classical

\[
\begin{align*}
\Gamma, \alpha & \vdash \Delta \\
\Gamma & \vdash \beta, \Delta \\
\Gamma & \vdash \neg \alpha, \Delta \\
\Gamma & \vdash \neg \beta \vdash \Delta \\
\end{align*}
\]
The logic and the variety

Curios

Let $L$ be the logic defined above.

1. $L$ is not algebraizable in the sense of Blok-Pigozzi.
2. $L$ is not order-algebraizable in the sense of Raftery.
3. $L$ is algebraizable as a sequent system, in the sense of Rebagliato-Verdú and Blok-Jónsson. Thus, BDLN is a natural semantics of $L$.
4. BDLN is not point-regular.
5. BDLN has the finite embeddability property.
6. The lattice reduct of the free zero-generated algebra in BDLN is a chain has order type $\omega + \omega^*$.
7. Cut elimination holds in $L$. 
The dual category: objects

**Definition**

The objects are pairs \((P, \mathcal{N} : P \to \mathcal{O}(\text{ClopUp}(P)))\), where

1. \(P\) is a Priestley space.
2. \(\text{ClopUp}(P)\) is the set of clopen up-sets of \(P\).
3. \(\mathcal{O}(\text{ClopUp}(P))\) is the set of downsets of \(\text{ClopUp}(P)\).
4. \(\mathcal{N} : P \to \mathcal{O}(\text{ClopUp}(P))\) is an order-preserving map, such that for every \(X \in \text{ClopUp}(P)\), the set \(\{p \in P : X \in \mathcal{N}(p)\}\) is clopen.

- \(\{p \in P : X \in \mathcal{N}(p)\}\) will be \(\neg X\).
- If \(P\) is finite, then \(\text{ClopUp}(P)\) is just the set of up-sets of \(P\), and (4) is satisfied by any order-preserving map.
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▶ \(\{p \in P : X \in \mathcal{N}(p)\}\) will be \(\neg X\).

▶ If \(P\) is finite, then \(\text{ClopUp}(P)\) is just the set of up-sets of \(P\), and (4) is satisfied by any order-preserving map.

▶ Example: the simplest that can be...
The dual category: preparing for morphisms

- Any order-preserving map \( h: P \rightarrow Q \) between ordered sets \( P \) and \( Q \) can be naturally lifted to a map \( h^{-1}: \mathcal{P}(Q) \rightarrow \mathcal{P}(P) \) taking each \( X \in \mathcal{P}(Q) \) to \( h^{-1}(X) \in \mathcal{P}(P) \).

- \( h^{-1} \) maps up-sets to up-sets and downsets to downsets.

- The lifting can be iterated. E.g., \( (h^{-1})^{-1}: \mathcal{P}(\mathcal{P}(P)) \rightarrow \mathcal{P}(\mathcal{P}(Q)) \). We will write \( \overline{h} \) for this double lifting.

- \( \overline{h} \) maps up-sets to up-sets and downsets to downsets.

- Let \( (P, \mathcal{N}^P) \) and \( (Q, \mathcal{N}^Q) \) be objects, and let \( h: P \rightarrow Q \) be a continuous map. Since \( h \) is continuous, the map \( h^{-1}: \text{ClopUp}(Q) \rightarrow \text{ClopUp}(P) \) is well defined.

- Thus, \( \overline{h} \) is also well defined as a map from \( \mathcal{O}(\text{ClopUp}(P)) \) to \( \mathcal{O}(\text{ClopUp}(Q)) \).
The dual category: morphisms

- Let $h: P \to Q$ be a continuous order-preserving map. Then, for any $W \in \mathcal{O}(\text{ClopUp}(P))$, we have 
  \[ \bar{h}(W) = \{ U \in \text{ClopUp}(Q): h^{-1}(U) \in W \}. \]

Definition

A morphism from $(P, \mathcal{N}^P)$ to $(Q, \mathcal{N}^Q)$ is a continuous order-preserving map $h: P \to Q$ such that the diagram below commutes.

\[
\begin{array}{ccc}
P & \xrightarrow{h} & Q \\
\mathcal{N}^P & \downarrow & \mathcal{N}^Q \\
\mathcal{O}(\text{ClopUp}(P)) & \xrightarrow{\bar{h}} & \mathcal{O}(\text{ClopUp}(Q))
\end{array}
\]
Dual equivalence

**Theorem**

*The categories BDLN (with homomorphisms) and OTNS are dually equivalent.*

Define $E : \text{OTNS} \to \text{BDLN}$ as follows:

- For an object $\mathcal{P} \in \text{OTNS}$, we put
  
  $$E(\mathcal{P}) = \left( \text{ClopUp}(\mathcal{P}), \cup, \cap, \neg, \emptyset, \mathcal{P} \right)$$

  where for every $X \in \text{ClopUp}(\mathcal{P})$ we have
  
  $$\neg X = \{ p \in \mathcal{P} : X \in \mathcal{N}(p) \}.$$ 

- For a morphism $h \in \text{Hom}(\mathcal{P}, \mathcal{Q})$, we put
  
  $$E(h)(U) = h^{-1}(U)$$

  for every $U \in \text{ClopUp}(\mathcal{P})$. 

Dual equivalence

Define $D: \text{BDLN} \rightarrow \text{OTNS}$, as follows:

- For an algebra $A \in \text{BDLN}$, we first take the usual Priestley topology on the set $\mathcal{F}_p(A)$ of all prime filters of $A$, and then, we put

$$D(A) = \left( \mathcal{F}_p(A), \mathcal{N}_A : \mathcal{F}_p(A) \rightarrow \mathcal{O}(\text{ClopUp}(\mathcal{F}_p(A))) \right)$$

where for every $F \in \mathcal{F}_p(A)$ we have

$$\mathcal{N}_A(F) = \left\{ \{H \in \mathcal{F}_p(A) : a \in H\} : \neg a \in F \right\}.$$

- For a homomorphism $f \in \text{Hom}(A, B)$, we put

$$D(f) = f^{-1}$$

where $D(f)(G) = f^{-1}(G)$ for every $G \in \mathcal{F}_p(B)$. 
Frame conditions

- Some examples of conditions on the algebras and corresponding conditions on dual spaces. Such things are known as frame conditions in dualities for BAOs.

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Dual space</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\neg 1 = 0$</td>
<td>$\forall p \in P: P \notin \mathcal{N}(p)$</td>
</tr>
<tr>
<td>$\neg 0 = 1$</td>
<td>$\forall p \in P: P \notin \mathcal{N}(p)$</td>
</tr>
<tr>
<td>$\neg x$ is the pseudo-complement of $x$</td>
<td>$X \in \mathcal{N}(p)$ iff $\uparrow p \cap X = \emptyset$</td>
</tr>
<tr>
<td>$\neg$ is a dual endomorphism</td>
<td>$\forall p \in P: \mathcal{N}(p) \in \text{im}(P)$</td>
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where $\text{im}(P)$ is the image of $P$ under the natural order-embedding of $P$ into $\mathcal{O}(\mathcal{U}(P))$. 
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<table>
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<td>¬1 = 0</td>
<td>∀p ∈ P: P ∉ N(p)</td>
</tr>
<tr>
<td>¬0 = 1</td>
<td>∀p ∈ P: P ∉ N(p)</td>
</tr>
<tr>
<td>¬x is the pseudo-complement of x</td>
<td>X ∈ N(p) iff ↑p ∩ X = ∅</td>
</tr>
<tr>
<td>¬ is a dual endomorphism</td>
<td>∀p ∈ P: N(p) ∈ im(P)</td>
</tr>
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where \( im(P) \) is the image of \( P \) under the natural order-embedding of \( P \) into \( \mathcal{O}(\mathcal{U}(P)) \).

- The third condition corresponds to an intuitionistic negation, the fourth to a de Morgan negation (the algebras are known as Ockham lattices).
Lattice of subvarieties

- Level 1. There are 3 atoms: generated by the 3 algebras based on the 2-element chain.
- Level 2. Algebras based on the 3-element chain generate 5 more join-irreducible varieties (there are 3 more: varietal joins of the atoms).