A multi-valued framework for coalgebraic logics over generalised metric spaces

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Motivation

Coalgebras encompass a wide variety of dynamical systems.

Their behaviour can be universally characterised using the theory of coalgebras.

However, in real life, the complexity of dynamical systems often makes bisimilarity is a too strict concept.

Consequently, the focus should be on **quantitative** behaviour (e.g. ordered, fuzzy, or probabilistic behavior):

(bi)similarity **pseudometric** that measures how similar two systems are from the point of view of their behaviours

These can be properly captured using coalgebras based on **quantale-enriched categories**.

Coalgebras and their logics – the abstract recipe

The coalgebraic data:

- Category $\mathcal{C}$
- Functor $T : \mathcal{C} \to \mathcal{C}$

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$$\begin{tikzcd} & \mathrm{Coalg}(T) \arrow[d] \\ & \mathcal{C} \arrow[loop left, "T"] \end{tikzcd}$$

- Category $\mathcal{C}$
- Functor $T : \mathcal{C} \to \mathcal{C}$
- T-coalgebra
  $c : X \to TX$
- T-coalgebra morphism

  $f : (X, c) \to (X', c')$

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "f"'] & TX \arrow[d, "Tf"] \\ X' \arrow[r, "c'"] & TX' \end{tikzcd}$$

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& \mathrm{Coalg}(T)^{\mathrm{op}} \arrow[d] & \\
& \mathcal{C}^{\mathrm{op}} \arrow[loop left, "T^{\mathrm{op}}"] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left]
\end{tikzcd}$$

The logical data:

- Contravariant adjunction

  $S \dashv P : \mathcal{D} \to \mathcal{C}^{\mathrm{op}}$

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- Contravariant adjunction

  $S \dashv P : \mathcal{D} \to \mathcal{C}^{\mathrm{op}}$
- Functor $L : \mathcal{D} \to \mathcal{D}$
- Natural transformation

  $\delta : LP \to PT^{op}$

This talk

Today's purpose: to look for a contravariant adjunction (to be used in the future for logics) for coalgebras over quantale-enriched categories.

Let $\mathscr{V}$ denote a commutative integral quantale.

Let $\mathscr{V}$-cat be the category of $\mathscr{V}$-categories and $\mathscr{V}$-functors.

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An ideal picture:

$$
\begin{tikzcd}
\text{base for coalgebras} \arrow[r, rightsquigarrow] & \mathsf{Spaces}^{\mathsf{op}} \arrow[r, shift left=2] \arrow[d] \arrow[r, phantom, "\top"] & \mathsf{Algebras} \arrow[l, shift left=2] \arrow[d] & \text{base for logics} \arrow[l, rightsquigarrow] \\
 & \mathscr{V}\text{-}\mathsf{cat}^{\mathsf{op}} \arrow[r, shift left=2, "{[-,\mathscr{V}]}"] \arrow[r, phantom, "\top"] & \mathscr{V}\text{-}\mathsf{cat} \arrow[l, shift left=2, "{[-,\mathscr{V}]}"] &
\end{tikzcd}
$$

▸ Coalgebraic side

▸ Logical side

- For $\mathscr{V} = \mathbb{2}$, this is relatively well understood.
- What about for other quantale $\mathscr{V}$? For $\mathscr{V} = ([0,1], \otimes, 1)$, for example?

A hint from positive coalgebraic logics

- The simplest case: the quantale $\mathbb{2}$

$$\mathsf{Poset}^{\mathsf{op}} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathsf{DLat}$$

  - Posets: antisymmetric $\mathbb{2}$-enriched categories.
  - Distributive lattices: antisymmetric finitely complete and cocomplete $\mathbb{2}$-categories such that finite limits distribute over finite colimits.

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- Move from $\mathbb{2}$ to an arbitrary quantale $\mathscr{V}$ – a naive approach:
  - Replace posets by antisymmetric $\mathscr{V}$-categories.
  - Replace distributive lattices by finitely complete and cocomplete $\mathscr{V}$-categories such that finite **conical** limits distribute over finite **conical** colimits.

- **Does it work?** A minimal requirement: the quantale $\mathscr{V}$ itself should have a distributive lattice reduct.

The contravariant adjunction – step I

- Consider the finitely complete and cocomplete $\mathscr{V}$-categories such that

$$\text{finite conical limits distribute over finite conical colimits} \qquad (\star)$$

with left and right exact $\mathscr{V}$-functors between them.

- Recall that finite colimits/limits can be completely described in terms of tensors/cotensors and finite joins/meets with respect to the underlying order of a $\mathscr{V}$-category.

- Hence each $\mathscr{A}$ as above is in particular a distributive lattice by $(\star)$, and each $f : \mathscr{A} \to \mathscr{Y}$ left and right exact is a morphism of distributive lattices.

- Tensors and cotensors are encoded by a family of adjoint pair of maps $r \odot - \dashv \pitchfork(r, -)$ on the underlying distributive lattice of $\mathscr{A}$, and lex/rex $\mathscr{V}$-functors preserve them.

The contravariant adjunction – step I

- In view of the previous features, call the resulting structure a distributive lattice with $\mathscr{V}$-operators (dlao($\mathscr{V}$)). In detail:
  - $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice.
  - A is endowed with a family of adjoint maps

    $$r \odot - \dashv \pitchfork(r, -) : A \to A \ , \ \ r \in \mathscr{V}$$

    satisfying the following:
    - $1 \odot a = a$
    - $(r \otimes r') \odot a = r \odot (r' \odot a)$
    - For each family $(r_i)_{i \in I}$ in $\mathscr{V}$ with $\bigvee_{i \in I} r_i = r$,

      $$r_i \odot a \leq r \odot a \quad \forall\, i \in I$$
      $$r_i \odot a \leq b \ , \quad \forall\, i \in I \implies r \odot a \leq b$$

- Morphisms of dlao($\mathscr{V}$) are those preserving all operations.
- Hence we obtain a category DLatAO($\mathscr{V}$) (more precisely a $\mathscr{V}$-cat-category)

The contravariant adjunction – step II

- The dual of DLatAO($\mathscr{V}$) can be obtained by restricted Priestley duality:

The contravariant adjunction – step II

- The dual of $\mathsf{DLatAO}(\mathscr{V})$ can be obtained by restricted Priestley duality:
- Objects of $\mathsf{DLatAO}(\mathscr{V})^{\mathsf{op}}$ are Priestley spaces $(X, \tau, \leq)$, endowed with a family of binary relations $(R_r)_{r \in \mathscr{V}}$ satisfying
  - $x' \leq x$ and $R_r(x,y)$ and $y \leq y'$ imply $R_r(x', y')$
  - $R_1 = \leq$
  - $R_r \circ R_{r'} = R_{r \otimes r'}$
  - $R_{\bigvee_{i \in I} r_i} = \bigcup_{i \in I} R_{r_i}$

  and several topological conditions.

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  and several topological conditions.
- Morphisms in $\mathsf{DLatAO}(\mathscr{V})^{\mathrm{op}}$ are monotone continuous maps $f : X \to Y$ such that
  - $R_r(x, y) \implies R_r(fx, fy)$
  - $R_r(u, fx) \iff (\exists x' \in X \,.\, u \leq fx' \text{ and } R_r(x', x))$
  - $R_r(fx, u) \iff (\exists x' \in X \,.\, R_r(x, x') \text{ and } fx' \leq u)$

The contravariant adjunction – step II

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- Assume that $\mathscr{V}$ is completely distributive and recall that each relational Priestley space is in particular compact Hausdorff.
- For completely distributive $\mathscr{V}$, the $\mathscr{V}$-cat-ification $\mathbb{U}^{\mathscr{V}}$ of the ultrafilter monad is a monad on $\mathscr{V}$-cat, hence we may speak of compact $\mathscr{V}$-categories as $\mathbb{U}^{\mathscr{V}}$-algebras.
- The $\mathscr{V}$-category structure and the compact Hausdorff structure on X are compatible, in the sense that the convergence map assigning to each ultrafilter on X its limit point is a $\mathscr{V}$-functor.

Conclusion

$$\mathsf{RelPriest}(\mathscr{V})^{\mathrm{op}} \cong \mathsf{DLatAO}(\mathscr{V})$$

compact Hausdorff $\mathscr{V}$-categories, hence antisymmetric and Cauchy complete	finitely complete and cocomplete $\mathscr{V}$-categories with distributivity condition $(\star)$

- We have obtained a duality between spaces and algebras, both carrying underlying $\mathscr{V}$-category structure.
- The duality above is yet unsatisfactory, as it is not obtained by "homming" into $\mathscr{V}$.
- In [Băbuș&Kurz'16], a duality between completely distributive $\mathscr{V}$-categories and atomic Cauchy complete $\mathscr{V}$-categories was provided. How are the two dualities related?

Thank you for your attention!

On the structure of $\mathscr{V}$-categories

[B, Kurz, Velebil – CALCO2015]

Fact: The discrete functor $D : \mathsf{Set} \to \mathscr{V}\text{-cat}$ is **dense**: each $\mathscr{V}$-category can be canonically expressed as a colimit of discrete ones.

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First, notice that each $\mathscr{V}$-category $\mathscr{A}$ determines the following data:

- A, the underlying *set* of objects of the $\mathscr{V}$-category $\mathscr{A}$
- $A_r = \{(a, b) \in A \times A \mid r \leq \mathscr{A}(a, b)\}$, the r-level set
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For each category $\mathscr{A}$, the above data can be organised as to describe a diagram (a $\mathscr{V}$-cat-functor) $F_{\mathscr{A}} : \mathrm{N} \to \mathrm{Set}$, hence a diagram of **discrete** $\mathscr{V}$-categories

$$\mathrm{N} \overset{F_{\mathscr{A}}}{\to} \mathrm{Set} \overset{D}{\to} \mathscr{V}\text{-cat}$$

Then the colimit of $DF_{\mathscr{A}}$ weighted by a convenient fixed presheaf ϕ is $\mathscr{A}$.

On functors for $\mathscr{V}$-cat-coalgebras

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In order to understand endofunctors (and their coalgebras) on $\mathscr{V}$-cat, look first at endofunctors on Set, then ask:

How to move from Set **to** $\mathscr{V}$-cat**?**

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Fact: Functors $T : \mathsf{Set} \to \mathsf{Set}$ can be canonically extended to $\mathscr{V}$-cat-functors $T_{\mathscr{V}} : \mathscr{V}\text{-cat} \to \mathscr{V}\text{-cat}$.

$$\begin{array}{ccc} \mathsf{Set} & \xrightarrow{D} & \mathscr{V}\text{-cat} \\ {\scriptstyle T}\downarrow & & \vdots\, {\scriptstyle T_{\mathscr{V}}} \\ \mathsf{Set} & \xrightarrow{D} & \mathscr{V}\text{-cat} \end{array}$$

Here canonically means $T_{\mathscr{V}} = \mathrm{Lan}_D(DT)$.

Call $T_{\mathscr{V}}$ the $\mathscr{V}$-cat-**ification** of T.

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$$\begin{tikzcd} \mathsf{Set} \arrow[r, "D"] \arrow[d, "T"'] & \mathscr{V}\text{-cat} \arrow[d, dotted, "T_{\mathscr{V}}"] \\ \mathsf{Set} \arrow[r, "D"] & \mathscr{V}\text{-cat} \end{tikzcd}$$

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How? The construction of the extension applies DT to the "$\mathscr{V}$-nerve" $F_{\mathscr{A}}$ of a $\mathscr{V}$-category $\mathscr{A}$, and then takes the appropriate "quotient" (colimit) $\phi \star (DTF\mathscr{A})$.

On functors for $\mathscr{V}$-cat-coalgebras

[B, Kurz, Velebil – CALCO2015]

An easier recipe: if T preserves weak pullbacks, then its $\mathscr{V}$-cat-ification can be computed using Barr's relation lifting.

$$T_{\mathscr{V}}\mathscr{A}(\mathfrak{a},\mathfrak{b}) = \bigvee_r \{r \mid (\mathfrak{a},\mathfrak{b}) \in \mathrm{Rel}_T(A_r)\}$$

Example: for $\mathscr{V}$ completely distributive, the $\mathscr{V}$-cat-ification of the powerset functor $\mathcal{P}$ gives the familiar Pompeiu-Hausdorff metric:

Let $\mathscr{A}$ be a $\mathscr{V}$-category. Then $\mathcal{P}_{\mathscr{V}}\mathscr{A}$ is the $\mathscr{V}$-category with objects $\mathcal{P}X$, and $\mathscr{V}$-homs

$$\mathcal{P}_{\mathscr{V}}\mathscr{A}(\mathfrak{a},\mathfrak{b}) = \big(\bigwedge_{a\in\mathfrak{a}}\bigvee_{b\in\mathfrak{b}}\mathscr{A}(a,b)\big)\bigwedge\big(\bigwedge_{b\in\mathfrak{b}}\bigvee_{a\in\mathfrak{a}}\mathscr{A}(a,b)\big)$$

We can even do better: **completely characterise** the $\mathscr{V}$-cat-endofunctors which arise as $\mathscr{V}$-cat-ifications, namely as being those which preserve all colimits $\phi\star DF_{\mathscr{A}}$ and also the discrete $\mathscr{V}$-categories.