

A multi-valued framework for coalgebraic logics over generalised metric spaces

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Coalgebras encompass a wide variety of dynamical systems.

Their behaviour can be universally characterised using the theory of coalgebras.

However, in real life, the complexity of dynamical systems often makes bisimilarity is a too strict concept.

Consequently, the focus should be on **quantitative** behaviour (e.g. ordered, fuzzy, or probabilistic behavior):

(bi)similarity **pseudometric** that measures how similar two systems are from the point of view of their behaviours

These can be properly captured using coalgebras based on **quantale-enriched categories**.

Coalgebras and their logics - the abstract recipe



The coalgebraic data:

- ▶ Category C
- Functor $T : C \to C$



Coalgebras and their logics - the abstract recipe



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- T-coalgebra $c: X \to TX$
- ► *T*-coalgebra morphism
 - $f:(X,c)\to (X',c')$



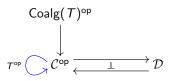
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The logical data:

- Contravariant adjunction
 - $S\dashv P:\mathcal{D}\to \mathcal{C}^{\mathsf{op}}$



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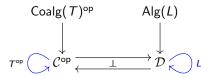
A multi-valued framework for coalgebraic logics

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Coalgebras and their logics - the abstract recipe

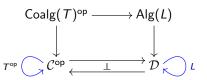
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 $\begin{array}{c} X \xrightarrow{c} TX \\ f \downarrow & \downarrow Tf \\ X' \xrightarrow{c'} TX' \end{array}$

The logical data:

- Contravariant adjunction
 - $S\dashv P:\mathcal{D}\to \mathcal{C}^{\mathsf{op}}$
- Functor $L : \mathcal{D} \to \mathcal{D}$
- Natural transformation
 - $\delta: LP \to PT^{op}$





This talk



Today's purpose: to look for a contravariant adjunction (to be used in the future for logics) for coalgebras over quantale-enriched categories.

Let $\mathscr V$ denote a commutative integral quantale.

Let $\mathscr{V}\text{-}\mathsf{cat}$ be the category of $\mathscr{V}\text{-}\mathsf{categories}$ and $\mathscr{V}\text{-}\mathsf{functors}.$

This talk

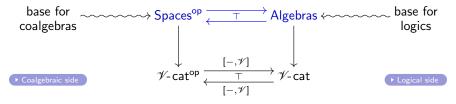


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An ideal picture:



• For $\mathscr{V} = 2$, this is relatively well understood.

• What about for other quantale \mathscr{V} ? For $\mathscr{V} = ([0,1],\otimes,1)$, for example?

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A hint from positive coalgebraic logics



 \blacktriangleright The simplest case: the quantale 2



- ▶ Posets: antisymmetric 2-enriched categories.
- ► Distributive lattices: antisymmetric finitely complete and cocomplete 2-categories such that finite limits distribute over finite colimits.

A hint from positive coalgebraic logics



 \blacktriangleright The simplest case: the quantale 2



- ▶ Posets: antisymmetric 2-enriched categories.
- ► Distributive lattices: antisymmetric finitely complete and cocomplete 2-categories such that finite limits distribute over finite colimits.
- Move from 2 to an arbitrary quantale \mathscr{V} a naive approach:
 - Replace posets by antisymmetric \mathscr{V} -categories.
 - Replace distributive lattices by finitely complete and cocomplete *V*-categories such that finite conical limits distribute over finite conical colimits.
- ► Does it work? A minimal requirement: the quantale *V* itself should have a distributive lattice reduct.

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The contravariant adjunction - step I



(*)

 \blacktriangleright Consider the finitely complete and cocomplete $\mathscr V\text{-categories}$ such that

finite conical limits distribute over finite conical colimits

with left and right exact \mathscr{V} -functors between them.

- Recall that finite colimits/limits can be completely described in terms of tensors/cotensors and finite joins/meets with respect to the underlying order of a *V*-category.
- ► Hence each A as above is in particular a distributive lattice by (*), and each f : A → Y left and right exact is a morphism of distributive lattices.
- ▶ Tensors and cotensors are encoded by a family of adjoint pair of maps $r \odot \dashv \pitchfork(r, -)$ on the underlying distributive lattice of \mathscr{A} , and lex/rex \mathscr{V} -functors preserve them.

The contravariant adjunction - step I



- In view of the previous features, call the resulting structure a distributive lattice with 𝒴-operators (dlao(𝒴)). In detail:
 - $(A, \land, \lor, 0, 1)$ is a bounded distributive lattice.
 - A is endowed with a family of adjoint maps

$$r \odot - \dashv \pitchfork (r, -) : A \rightarrow A \ , \ r \in \mathscr{V}$$

satisfying the following:

$$(r \otimes r') \odot a = r \odot (r' \odot a)$$

► For each family $(r_i)_{i \in I}$ in \mathscr{V} with $\bigvee_{i \in I} r_i = r$, $r_i \odot a \le r \odot a \quad \forall i \in I$ $r_i \odot a \le b$, $\forall i \in I \implies r \odot a \le b$

- Morphisms of dlao(\u03c6) are those preserving all operations.
- Hence we obtain a category DLatAO(\mathscr{V}) (more precisely a \mathscr{V} -cat-category)

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The contravariant adjunction - step II



• The dual of DLatAO(\mathscr{V}) can be obtained by restricted Priestley duality:

The contravariant adjunction – step II



- The dual of DLatAO(\mathscr{V}) can be obtained by restricted Priestley duality:
- Objects of DLatAO(𝒴)^{op} are Priestley spaces (X, τ, ≤), endowed with a family of binary relations (R_r)_{r∈𝒴} satisfying
 - $x' \leq x$ and $R_r(x, y)$ and $y \leq y'$ imply $R_r(x', y')$

•
$$R_1 = \leq$$

$$\blacktriangleright R_r \circ R_{r'} = R_{r \otimes r'}$$

•
$$R_{\bigvee_{i\in I}r_i} = \bigcup_{i\in I}R_{r_i}$$

and several topological conditions.

The contravariant adjunction – step II



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and several topological conditions.

• Morphisms in DLatAO(\mathscr{V})^{op} are monotone continuous maps $f: X \to Y$ such that

$$\blacktriangleright R_r(x,y) \Longrightarrow R_r(fx,fy)$$

- $R_r(u, fx) \iff (\exists x' \in X . u \le fx' \text{ and } R_r(x', x))$
- $R_r(fx, u) \iff (\exists x' \in X . R_r(x, x') \text{ and } fx' \leq u)$

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The contravariant adjunction - step II



• Denote by $RelPriest(\mathscr{V})$ the resulting category. Hence

 $\mathsf{RelPriest}(\mathscr{V})^{\mathsf{op}} \cong \mathsf{DLatAO}(\mathscr{V})$

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The contravariant adjunction – step II



• Denote by $RelPriest(\mathcal{V})$ the resulting category. Hence

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▶ Each relational Priestley space X becomes a \mathscr{V} -category by

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- ► Assume that 𝒴 is completely distributive and recall that each relational Priestley space is in particular compact Hausdorff.
- For completely distributive 𝒴, the 𝒴-cat-ification 𝔅𝑘𝒱 of the ultrafilter monad is a monad on 𝒴-cat, hence we may speak of compact 𝒴-categories as 𝔅𝑘𝑘𝑘 -algebras.
- ► The V-category structure and the compact Hausdorff structure on X are compatible, in the sense that the convergence map assigning to each ultrafilter on X its limit point is a V-functor.

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Conclusion



 $\begin{array}{ccc} \mathsf{RelPriest}(\mathscr{V})^{\mathsf{op}} &\cong &\mathsf{DLatAO}(\mathscr{V}) \\ &&&& \\ &&&& \\ &&&& \\ &&&& \\ &&&& \\ &&&& \\ &&$

- ► We have obtained a duality between spaces and algebras, both carrying underlying *V*-category structure.
- ► The duality above is yet unsatisfactory, as it is not obtained by "homming" into 𝒴.
- In [Băbuş&Kurz'16], a duality between completely distributive 𝒴-categories and atomic Cauchy complete 𝒴-categories was provided. How are the two dualities related?

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Thank you for your attention!

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On the structure of \mathscr{V} -categories [B, Kurz, Velebil – CALCO2015]



Fact: The discrete functor D: Set $\rightarrow \mathscr{V}$ -cat is **dense**: each \mathscr{V} -category can be canonically expressed as a colimit of discrete ones.

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First, notice that each $\mathscr V\text{-category}\ \mathscr A$ determines the following data:

- ► *A*, the underlying *set* of objects of the *V*-category *A*
- $A_r = \{(a, b) \in A \times A \mid r \leq \mathscr{A}(a, b)\}$, the *r*-level set
- $d_0^r, d_1^r: A_r \to A$ the usual projection maps

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For each category \mathscr{A} , the above data can be organised as to describe a diagram (a \mathscr{V} -cat-functor) $F_{\mathscr{A}} : \mathbb{N} \to \operatorname{Set}$, hence a diagram of **discrete** \mathscr{V} -categories

$$\mathsf{N} \stackrel{\mathcal{F}_{\mathscr{A}}}{\to} \mathsf{Set} \stackrel{D}{\to} \mathscr{V}\text{-}\mathsf{cat}$$

Then the colimit of $DF_{\mathscr{A}}$ weighted by a convenient fixed presheaf ϕ is \mathscr{A} .

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On functors for \mathscr{V} -cat-coalgebras

[B, Kurz, Velebil - CALCO2015]



In order to understand endofunctors (and their coalgebras) on \mathscr{V} -cat, look first at endofunctors on Set, then ask:

How to move from Set to *Y*-cat?

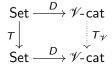
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Here canonically means $T_{\mathscr{V}} = \operatorname{Lan}_D(DT)$.

Call $T_{\mathscr{V}}$ the \mathscr{V} -cat-ification of T.



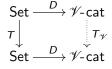
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Call $T_{\mathscr{V}}$ the \mathscr{V} -cat-ification of T.

How? The construction of the extension applies DT to the " \mathscr{V} -nerve" $F_{\mathscr{A}}$ of a \mathscr{V} -category \mathscr{A} , and then takes the appropriate "quotient" (colimit) $\phi \star (DTF\mathscr{A})$.

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On functors for \mathscr{V} -cat-coalgebras [B, Kurz, Velebil – CALCO2015]



An easier recipe: if T preserves weak pullbacks, then its \mathscr{V} -cat-ification can be computed using Barr's relation lifting.

$$T_{\mathscr{V}}\mathscr{A}(\mathfrak{a},\mathfrak{b}) = \bigvee_{r} \{r \mid (\mathfrak{a},\mathfrak{b}) \in \operatorname{Rel}_{T}(A_{r})\}$$

Example: for \mathscr{V} completely distributive, the \mathscr{V} -cat-ification of the powerset functor \mathcal{P} gives the familiar Pompeiu-Hausdorff metric:

Let \mathscr{A} be a \mathscr{V} -category. Then $\mathcal{P}_{\mathscr{V}}\mathscr{A}$ is the \mathscr{V} -category with objects $\mathcal{P}X$, and \mathscr{V} -homs

$$\mathcal{P}_{\mathscr{V}}\mathscr{A}(\mathfrak{a},\mathfrak{b}) = ig(\bigwedge_{a\in\mathfrak{a}} \bigvee_{b\in\mathfrak{b}} \mathscr{A}(a,b) ig) ig) ig(\bigwedge_{b\in\mathfrak{b}} \bigvee_{a\in\mathfrak{a}} \mathscr{A}(a,b) ig)$$

We can even do better: **completely characterise** the \mathscr{V} -cat-endofunctors which arise as \mathscr{V} -cat-ifications, namely as being those which preserve all colimits $\phi \star DF_{\mathscr{A}}$ and also the discrete \mathscr{V} -categories.