A multi-valued framework for coalgebraic logics over generalised metric spaces

Adriana Balan
University Politehnica of Bucharest

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Motivation

Coalgebras encompass a wide variety of dynamical systems.

Their behaviour can be universally characterised using the theory of coalgebras.

However, in real life, the complexity of dynamical systems often makes bisimilarity a too strict concept.

Consequently, the focus should be on quantitative behaviour (e.g. ordered, fuzzy, or probabilistic behavior):

(bi)similarity pseudometric that measures how similar two systems are from the point of view of their behaviours.

These can be properly captured using coalgebras based on quantale-enriched categories.
Coalgebras and their logics – the abstract recipe

The coalgebraic data:

- Category \( C \)
- Functor \( T : C \to C \)

\[ \text{Coalg}(T) \]

\[ \text{Alg}(L) \]

The logical data:

- Contravariant adjunction \( S \dashv P : D \to C \)
- Functor \( L : D \to D \)
- Natural transformation \( \delta : LP \to PT \)

\[ \text{op} \]
Coalgebras and their logics – the abstract recipe

The coalgebraic data:

- Category $\mathcal{C}$
- Functor $T : \mathcal{C} \to \mathcal{C}$
- $T$-coalgebra $c : X \to TX$
- $T$-coalgebra morphism $f : (X, c) \to (X', c')$

\[
\begin{array}{ccc}
X & \xrightarrow{c} & TX \\
\downarrow{f} & & \downarrow{Tf} \\
X' & \xrightarrow{c'} & TX'
\end{array}
\]
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- $T$-coalgebra morphism $f : (X, c) \to (X', c')$

The logical data:

- Contravariant adjunction $S \dashv P : \mathcal{D} \to \mathcal{C}^{\text{op}}$
Coalgebras and their logics – the abstract recipe

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- Category $C$
- Functor $T : C \to C$
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The logical data:

- Contravariant adjunction $S \dashv P : D \to C^{\text{op}}$
- Functor $L : D \to D$
Coalgebras and their logics – the abstract recipe

The coalgebraic data:

- **Category** \( C \)
- **Functor** \( T : C \to C \)
- **\( T \)-coalgebra** \( c : X \to TX \)
- **\( T \)-coalgebra morphism** \( f : (X, c) \to (X', c') \)

The logical data:

- **Contravariant adjunction** \( S \dashv P : D \to C^{\text{op}} \)
- **Functor** \( L : D \to D \)
- **Natural transformation** \( \delta : LP \to PT^{\text{op}} \)
This talk

Today’s purpose: to look for a *contravariant adjunction* (to be used in the future for logics) for coalgebras over quantale-enriched categories.

Let $\mathcal{V}$ denote a commutative integral quantale.

Let $\mathcal{V}$-cat be the category of $\mathcal{V}$-categories and $\mathcal{V}$-functors.
This talk

Today’s purpose: to look for a **contravariant adjunction** (to be used in the future for logics) for coalgebras over quantale-enriched categories.

Let \( \mathcal{V} \) denote a commutative integral quantale.

Let \( \mathcal{V}\text{-cat} \) be the category of \( \mathcal{V}\)-categories and \( \mathcal{V}\)-functors.

An ideal picture:

\[
\begin{array}{cccccc}
\text{base for} & \rightsquigarrow & \text{Spaces}^{\text{op}} & \xleftarrow{T} & \text{Algebras} & \leftarrow \text{base for} \\
\text{coalgebras} & \downarrow & \text{Spaces}^{\text{op}} & \xrightarrow{[-,\mathcal{V}]} & \mathcal{V}\text{-cat} & \xrightarrow{[-,\mathcal{V}]} \text{Algebras} \\
\mathcal{V}\text{-cat}^{\text{op}} & \xleftarrow{[-,\mathcal{V}]} & \mathcal{V}\text{-cat} & \text{logical side} \\
\end{array}
\]

- For \( \mathcal{V} = \mathcal{2} \), this is relatively well understood.
- What about for other quantale \( \mathcal{V} \)? For \( \mathcal{V} = ([0,1], \otimes, 1) \), for example?
The simplest case: the quantale \( \mathbb{2} \)

\[ \text{Poset}^{\text{op}} \xleftarrow{\perp} \text{DLat} \]

- Posets: antisymmetric \( \mathbb{2} \)-enriched categories.
- Distributive lattices: antisymmetric \textit{finitely complete and cocomplete} \( \mathbb{2} \)-categories such that \textit{finite limits distribute over finite colimits}. 
A hint from positive coalgebraic logics

- The simplest case: the quantale \( \mathcal{P} \)

\[
\text{Poset}^{\text{op}} \xleftarrow{\bot} \text{DLat}
\]

- Posets: antisymmetric \( \mathcal{P} \)-enriched categories.

- Distributive lattices: antisymmetric finitely complete and cocomplete \( \mathcal{P} \)-categories such that \text{finite limits distribute over finite colimits}.

- Move from \( \mathcal{P} \) to an arbitrary quantale \( \mathcal{V} \) – a naive approach:
  - Replace posets by antisymmetric \( \mathcal{V} \)-categories.
  - Replace distributive lattices by \text{finitely complete and cocomplete} \( \mathcal{V} \)-categories such that \text{finite conical limits distribute over finite conical colimits}.

- \textbf{Does it work?} A minimal requirement: the quantale \( \mathcal{V} \) itself should have a distributive lattice reduct.
The contravariant adjunction – step I

- Consider the finitely complete and cocomplete $\mathcal{V}$-categories such that

\[
\text{finite conical limits distribute over finite conical colimits} \quad (\star)
\]

with left and right exact $\mathcal{V}$-functors between them.

- Recall that finite colimits/limits can be completely described in terms of tensors/cotensors and finite joins/meets with respect to the underlying order of a $\mathcal{V}$-category.

- Hence each $\mathcal{A}$ as above is in particular a distributive lattice by (\star), and each $f : \mathcal{A} \to \mathcal{V}$ left and right exact is a morphism of distributive lattices.

- Tensors and cotensors are encoded by a family of adjoint pair of maps $r \odot - \dashv \sqcup (r, -)$ on the underlying distributive lattice of $\mathcal{A}$, and lex/lex $\mathcal{V}$-functors preserve them.
The contravariant adjunction – step I

- In view of the previous features, call the resulting structure a distributive lattice with \( \mathcal{V} \)-operators (dlao(\( \mathcal{V} \))). In detail:
  - \((A, \land, \lor, 0, 1)\) is a bounded distributive lattice.
  - \(A\) is endowed with a family of adjoint maps
    
    \[
    r \circ - \dashv \cap (r, -) : A \to A, \ r \in \mathcal{V}
    \]
    
    satisfying the following:
    - \(1 \circ a = a\)
    - \((r \otimes r') \circ a = r \circ (r' \circ a)\)
    - For each family \((r_i)_{i \in I}\) in \(\mathcal{V}\) with \(\bigvee_{i \in I} r_i = r\),
      
      \[
      r_i \circ a \leq r \circ a \quad \forall i \in I
      \]
      
      \[
      r_i \circ a \leq b, \quad \forall i \in I \implies r \circ a \leq b
      \]
  - Morphisms of dlao(\( \mathcal{V} \)) are those preserving all operations.
  - Hence we obtain a category DLatAO(\( \mathcal{V} \)) (more precisely a \( \mathcal{V} \)-cat-category)
The contravariant adjunction – step II

- The dual of $\text{DLatAO}(\mathcal{V})$ can be obtained by restricted Priestley duality:

Objects of $\text{DLatAO}(\mathcal{V})^{\text{op}}$ are Priestley spaces $(X, \tau, \leq)$, endowed with a family of binary relations $(R_r)_{r \in \mathcal{V}}$ satisfying:

1. $x \leq x'$ and $R_r(x, y)$ and $y \leq y'$ imply $R_r(x', y')$,
2. $R_1 \leq R_r \circ R_r' = R_r \otimes r'$,
3. $R_r \bigvee i \in I r_i = \bigcup i \in I R_{r_i}$ and several topological conditions.

Morphisms in $\text{DLatAO}(\mathcal{V})^{\text{op}}$ are monotone continuous maps $f : X \to Y$ such that:

1. $R_r(x, y) \Rightarrow R_r(fx, fy)$,
2. $R_r(fx, u) \Leftrightarrow (\exists x' \in X. u \leq fx' \text{ and } R_r(x', x))$,
3. $R_r(fx, u) \Leftrightarrow (\exists x' \in X. R_r(x, x') \text{ and } fx' \leq u)$. 

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The contravariant adjunction – step II

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  - \(x' \leq x\) and \(R_r(x, y)\) and \(y \leq y'\) imply \(R_r(x', y')\)
  
  - \(R_1 = \leq\)
  
  - \(R_r \circ R_{r'} = R_{r \otimes r'}\)
  
  - \(R_{\bigvee_{i \in I} r_i} = \bigcup_{i \in I} R_{r_i}\)

  and several topological conditions.
The contravariant adjunction – step II

- The dual of DLat\(\mathcal{A}O(\mathcal{V})\) can be obtained by restricted Priestley duality:

- Objects of DLat\(\mathcal{A}O(\mathcal{V})^{\text{op}}\) are Priestley spaces \((X, \tau, \leq)\), endowed with a family of binary relations \((R_r)_{r \in \mathcal{V}}\) satisfying
  - \(x' \leq x\) and \(R_r(x, y)\) and \(y \leq y'\) imply \(R_r(x', y')\)
  - \(R_1 = \leq\)
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and several topological conditions.

- Morphisms in DLat\(\mathcal{A}O(\mathcal{V})^{\text{op}}\) are monotone continuous maps \(f : X \to Y\) such that
  - \(R_r(x, y) \implies R_r(fx, fy)\)
  - \(R_r(u, fx) \iff (\exists x' \in X. u \leq fx' \text{ and } R_r(x', x))\)
  - \(R_r(fx, u) \iff (\exists x' \in X. R_r(x, x') \text{ and } fx' \leq u)\)
The contravariant adjunction – step II

Denote by $\text{RelPriest}(\mathcal{V})$ the resulting category. Hence

$$\text{RelPriest}(\mathcal{V})^{\text{op}} \cong \text{DLatAO}(\mathcal{V})$$
The contravariant adjunction – step II

- Denote by \( \text{RelPriest}(\mathcal{V}) \) the resulting category. Hence

\[
\text{RelPriest}(\mathcal{V})^{\text{op}} \cong \text{DLatAO}(\mathcal{V})
\]

- Each relational Priestley space \( X \) becomes a \( \mathcal{V} \)-category by

\[
\mathcal{X}(x, y) = \bigvee \{ r \mid R_r(x, y) \}
\]
The contravariant adjunction – step II

- Denote by $\text{RelPriest}(\mathcal{V})$ the resulting category. Hence
  \[
  \text{RelPriest}(\mathcal{V})^{\text{op}} \cong \text{DLatAO}(\mathcal{V})
  \]

- Each relational Priestley space $X$ becomes a $\mathcal{V}$-category by
  \[
  \mathcal{K}(x, y) = \bigvee \{r \mid R_r(x, y)\}
  \]

- Assume that $\mathcal{V}$ is completely distributive and recall that each relational Priestley space is in particular compact Hausdorff.

- For completely distributive $\mathcal{V}$, the $\mathcal{V}$-cat-ification $\mathbb{U} \mathcal{V}$ of the ultrafilter monad is a monad on $\mathcal{V}$-cat, hence we may speak of compact $\mathcal{V}$-categories as $\mathbb{U} \mathcal{V}$-algebras.

- The $\mathcal{V}$-category structure and the compact Hausdorff structure on $X$ are compatible, in the sense that the convergence map assigning to each ultrafilter on $X$ its limit point is a $\mathcal{V}$-functor.
Conclusion

We have obtained a duality between spaces and algebras, both carrying underlying \( V \)-category structure.

The duality above is yet unsatisfactory, as it is not obtained by “homming” into \( V \).

In [Băbuș&Kurz’16], a duality between completely distributive \( V \)-categories and atomic Cauchy complete \( V \)-categories was provided. How are the two dualities related?
Thank you for your attention!
Fact: The discrete functor $D : \text{Set} \to \mathcal{V}\text{-cat}$ is dense: each $\mathcal{V}$-category can be canonically expressed as a colimit of discrete ones.
On the structure of \( \mathcal{V} \)-categories

[B, Kurz, Velebil – CALCO2015]

**Fact:** The discrete functor \( D : \text{Set} \to \mathcal{V}\text{-cat} \) is **dense**: each \( \mathcal{V} \)-category can be canonically expressed as a colimit of discrete ones.

First, notice that each \( \mathcal{V} \)-category \( \mathcal{A} \) determines the following data:

- **A**, the underlying set of objects of the \( \mathcal{V} \)-category \( \mathcal{A} \)
- **\( A_r = \{(a, b) \in A \times A \mid r \leq \mathcal{A}(a, b)\} \)**, the \( r \)-level set
- **\( d_0^r, d_1^r : A_r \to A \)** the usual projection maps
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- $A$, the underlying set of objects of the $\mathcal{V}$-category $\mathcal{A}$
- $A_r = \{(a, b) \in A \times A \mid r \leq \mathcal{A}(a, b)\}$, the $r$-level set
- $d^r_0, d^r_1 : A_r \rightarrow A$ the usual projection maps

For each category $\mathcal{A}$, the above data can be organised as to describe a diagram (a $\mathcal{V}$-cat-functor) $F_{\mathcal{A}} : \mathbb{N} \rightarrow \text{Set}$, hence a diagram of discrete $\mathcal{V}$-categories

$$\begin{array}{ccc}
\mathbb{N} & \xrightarrow{F_{\mathcal{A}}} & \text{Set} \\
& & \downarrow D \\
& & \mathcal{V}\text{-cat}
\end{array}$$

Then the colimit of $DF_{\mathcal{A}}$ weighted by a convenient fixed presheaf $\phi$ is $\mathcal{A}$. 
On functors for $\mathcal{V}$-cat-coalgebras

[B, Kurz, Velebil – CALCO2015]

In order to understand endofunctors (and their coalgebras) on $\mathcal{V}$-cat, look first at endofunctors on Set, then ask:

**How to move from Set to $\mathcal{V}$-cat?**
On functors for $\mathcal{V}$-cat-coalgebras

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**Fact:** Functors $T : \text{Set} \to \text{Set}$ can be canonically extended to $\mathcal{V}$-cat-functors $T_\mathcal{V} : \mathcal{V}$-cat $\to \mathcal{V}$-cat.

Here canonically means $T_\mathcal{V} = \text{Lan}_D(DT)$. Call $T_\mathcal{V}$ the $\mathcal{V}$-cat-ification of $T$. 
On functors for $\mathcal{V}$-cat-coalgebras

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In order to understand endofunctors (and their coalgebras) on $\mathcal{V}$-cat, look first at endofunctors on Set, then ask:

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Call $T_\mathcal{V}$ the $\mathcal{V}$-cat-ification of $T$.

**How?** The construction of the extension applies $DT$ to the “$\mathcal{V}$-nerve” $F_A$ of a $\mathcal{V}$-category $A$, and then takes the appropriate “quotient” (colimit) $\phi \ast (DTF_A)$. 

![](image)
On functors for \( \mathcal{V} \text{-cat-coalgebras} \)

[B, Kurz, Velebil – CALCO2015]

**An easier recipe:** if \( T \) preserves weak pullbacks, then its \( \mathcal{V} \text{-cat-ification} \) can be computed using Barr’s relation lifting.

\[
T_{\mathcal{V} \mathcal{A}}(a, b) = \bigvee_r \{ r \mid (a, b) \in \text{Rel}_T(A_r) \}
\]

**Example:** for \( \mathcal{V} \) completely distributive, the \( \mathcal{V} \text{-cat-ification} \) of the powerset functor \( \mathcal{P} \) gives the familiar Pompeiu-Hausdorff metric:

Let \( \mathcal{A} \) be a \( \mathcal{V} \)-category. Then \( \mathcal{P}_{\mathcal{V} \mathcal{A}} \) is the \( \mathcal{V} \)-category with objects \( \mathcal{P}X \), and \( \mathcal{V} \)-homs

\[
\mathcal{P}_{\mathcal{V} \mathcal{A}}(a, b) = \left( \bigwedge_{a \in a} \bigvee_{b \in b} \mathcal{A}(a, b) \right) \bigwedge \left( \bigwedge_{b \in b} \bigvee_{a \in a} \mathcal{A}(a, b) \right)
\]

We can even do better: **completely characterise** the \( \mathcal{V} \)-cat-endofunctors which arise as \( \mathcal{V} \)-cat-ifications, namely as being those which preserve all colimits \( \phi \star DF \mathcal{A} \) and also the discrete \( \mathcal{V} \)-categories.