On Kripke completeness of modal and superintuitionistic predicate logics with equality

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Introduction

Unlike the propositional case, in first-order modal (and intuitionistic) logic there is a gap between syntax and semantics. It turns out that simply axiomatizable modal logics may have complex semantic descriptions. The standard Kripke semantics does not work properly in the predicate case - "most of" modal predicate logics are Kripke-incomplete.

As the semantics of predicate logics is not clearly understandable, natural questions about properties of logics may be quite difficult.

In this talk we consider only one issue:

adding equality to a predicate logic.

Reference

D.Gabbay, V. Shehtman, D. Skvortsov. Quantification in Nonclassical Logic, Volume 1. Elsevier, 2009.

Formulas

Intuitionistic predicate formulas are built from the following ingredients:

- the countable set of individual variables $Var=\{v_1,v_2,\ldots\}$
- countable sets of n-ary predicate letters (for every $n\geq 0$)
- $\rightarrow$, $\bot$, $\vee$, $\wedge$
- $\exists$, $\forall$

Modal predicate formulas can also contain $\Box$.

Formulas with equality can also contain $=$.

The connectives $\neg$, $\Diamond$ are derived.

No constants or function symbols

NOTATION for the sets of formulas: IF, $IF^{=}$, MF, $MF^{=}$

Variable and formula substitutions

$[y_1,\ldots, y_n /x_1,\ldots, x_n]$ simultaneously replaces all free occurrences of $x_1,\ldots, x_n$ with $y_1,\ldots, y_n$ (renaming bound variables if necessary)

To obtain $[C(x_1,\ldots, x_n,y_1,\ldots, y_m)/P(x_1,\ldots, x_n)]A$ from A

(1) rename all bound variables of A that coincide with the "new" parameters $y_1,\ldots, y_m$ of C,

(2) replace every occurrence of every atom $P(z_1,\ldots, z_n)$ with $[z_1,\ldots, z_n /x_1,\ldots, x_n]C$

Strictly speaking, all substitutions are defined up to congruence: formulas are congruent if they can be obtained by "legal" renaming of bound variables

$[Q(x,y,z)/P(x)]\ (\exists y P(y) \wedge P(z)) = \exists x Q(x,y,z) \wedge Q(z,y,z)$ or

$\exists u Q(u,y,z) \wedge Q(z,y,z)$

Modal logics

An modal predicate logic (mpl) is a set L of modal formulas such that L contains

- the classical propositional tautologies
- the axiom of **K:** $\Box(p \to q) \to (\Box p \to \Box q)$
- the standard predicate axioms

L is closed under the rules

- Modus Ponens: $A, A \to B \,/\, B$
- Necessitation: $A \,/\, \Box A$
- Generalization: $A \,/\, \forall x A$
- Substitution: A/SA (for any formula substitution S)

Superintuititionistic logics

A superintuitionistic predicate logic (spl) is a set L of intuitionistic formulas such that

- L contains the axioms of intuitionistic first-order logic **QH**
- L is closed under Modus Ponens
- L is closed under Generalization
- L is closed under (intuitionistic) formula substitutions

Modal/superintuitionistic logics with equality

An modal/superintuitionistic predicate logic with equality (mpl=/spl=) is a set of formulas with equality with the same properties as mpl/spl plus

1. Substitutions [C/x=y] are not allowed.
2. L contains the standard equality axioms.

Propositional logics can be regarded as fragments of predicate logics (with only 0-ary predicate letters, without quantifiers).

Some notation

$L+\Gamma$:= the smallest logic containing (L and Γ)

K := the minimal modal propositional logic

H := intuitionistic propositional logic

QL := the minimal predicate logic containing the propositional logic L

$L^{=}$:= the minimal logic with equality containing the predicate logic L (without equality), the equality-expansion of L

$L \vdash A$:= $A \in L$

<u>Def</u> A logic with equality L' is conservative over a logic without equality L (of the same type) if $L \subseteq L'$,

but for any A in the language of L, $L \vdash A$ iff $L' \vdash A$.

Kripke frame semantics for predicate logics

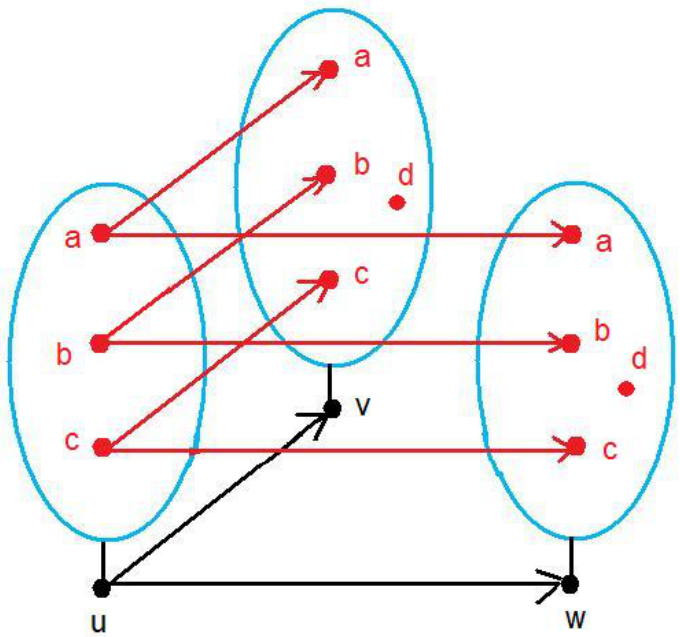
- A propositional Kripke frame $F=(W, R)$ $(W \neq \emptyset, R \subseteq W^2)$
- A predicate Kripke frame: $\Phi=(F,D)$, where

$D=(D_u)_{u \in W}$ is an expanding family of non-empty sets:

$$\text{if } u \, R \, v, \text{ then } D_u \subseteq D_v$$

D_u is the domain at the world u (consists of existing individuals).

- In intuitionistic frames R is reflexive transitive (or even a partial order)



A Kripke model over Φ is a collection of classical models:

$M=(\Phi,\theta)$, where $\theta=(\theta_u)_{u\in W}$ is a valuation

$\theta_u(P)$ is an n-ary relation on D_u for each n-ary predicate letter P

For every modal formula $A(x_1,\ldots, x_n)$ and $d_1,\ldots, d_n \in D_u$

consider a D_u-sentence $A(d_1,\ldots, d_n)$.

<u>Def</u> Forcing (truth) relation $M,u \models B$

between the worlds u and D_u-sentences B is defined by induction:

- $M,u \models P(d_1,\ldots, d_n)$ iff $(d_1,\ldots, d_n) \in \theta_u(P)$
- $M,u \models a=b$ iff a equals b
- $M,u \models \Box B$ iff for any v, uRv implies $M,v \models B$
- $M,u \models \forall x\, B$ iff for any $d \in D_u$ $M,u \models [d/x]B$

etc. (the other cases are clear)

Intuitionistic Kripke models

$M=(\Phi,\theta)$, where Φ is intuitionistic, $\theta=(\theta_u)_{u\in W}$ is a *stable* valuation: $uRv \Rightarrow \theta_u(P) \subseteq \theta_v(P)$

Forcing relation $M,u \Vdash B$

- $M,u \Vdash P(d_1,\ldots, d_n)$ iff $(d_1,\ldots, d_n) \in \theta_u(P)$
- $M,u \Vdash a=b$ iff a equals b
- $M,u \Vdash B \rightarrow C$ iff

  for any v, uRv & $M,v \Vdash B$ implies $M,v \Vdash C$
- $M,u \Vdash \exists x\, B$ iff for some $d \in D_u$ $M,u \Vdash [d/x]B$
- $M,u \Vdash \forall x\, B$ iff

  for any $v\in R(u)$ for any $d \in D_v$ $M,v \Vdash [d/x]B$

etc.

Def (truth in a Kripke model; validity in a frame)

$\mathrm{M} \vDash A(x_1,\ldots, x_n)$ iff for any $u \in W$ $\mathrm{M},u \vDash \forall x_1 \ldots \forall x_n A(x_1,\ldots, x_n)$

$\Phi \vDash_{\mathrm{A}}$ iff for any M over Φ, $\mathrm{M} \vDash A$

Similarly in the intuitionistic case.

Soundness theorem

(1) **ML**(Φ):$=\{A \in \mathrm{MF} \mid \Phi \vDash_{\mathrm{A}}\}$ is an mpl

(2) **ML**$^{=}(\Phi)$:$=\{A \in \mathrm{MF}^{=} \mid \Phi \vDash_{\mathrm{A}}\}$ is an mpl=

(3) **IL**(Φ):$=\{A \in \mathrm{IF} \mid \Phi \Vdash \mathrm{A}\}$ is an spl

(4) **IL**$^{=}(\Phi)$:$=\{A \in \mathrm{IF}^{=} \mid \Phi \Vdash \mathrm{A}\}$ is an spl=

Logics of this form are called *Kripke-complete*

Kripke frame semantics with equality

Kripke frames with equality (KFE)

$\Phi=(F,D,\approx)$, where

$\approx\ =(\approx_u)_{u\in W}$ is a family of expanding equivalence relations on the domains:

$$\text{if } u\ R\ v, \text{ then } \approx_u \subseteq \approx_v$$

Kripke models with equality should respect the equivalence relations:

if P is n-ary, $a_1,\ldots, a_n$, $b_1,\ldots, b_n$ are individuals and $(a_1,\ldots, a_n)\in\theta_u(P)$, $a_1\approx_u b_1, \ldots, a_n\approx_u b_n$, then

$$(b_1,\ldots, b_n)\in\theta_u(P)$$

The definition of forcing changes only for the equality:

$$M,u \models a=b \quad \text{iff} \quad a \approx_u b$$

<u>Soundness theorem</u> $\mathbf{ML}^{=}(\Phi):=\{A \in MF^{=} \mid \Phi \models A\}$ is an mpl=

and similarly for the intuitionistic case.

The standard Kripke frames can be regarded as KFEs, where all the $\approx_u$ are the identity relations.

Kripke sheaves

Kripke sheaves are an equivalent version of KFEs. They are obtained from KFEs by identifying equivalent individuals at every world.

<u>Def</u> A Kripke sheaf over a propositional reflexive transitive frame F=(W,R) is a triple $\mathscr{F}=(F,D,\rho)$, in which

(F,D) is a predicate Kripke frame, $\rho=(\rho_{uv})_{uRv}$ is a collection of *transition maps* ("cross-reference") $\rho_{uv}: D_u \to D_v$ such that:

ρ_{uu} is the identity function on D_u

if uRvRw, then ρ_{uw} is the composition $\rho_{vw}\,\rho_{uv}$

<u>Def.</u> For an arbitrary propositional frame $F=(W,R)$, consider the transitive reflexive closure $F^*:=(W, R^*)$ (uR^*v iff there is an oriented path from u to v in F:

$uRw_1 \ldots w_kRv$)

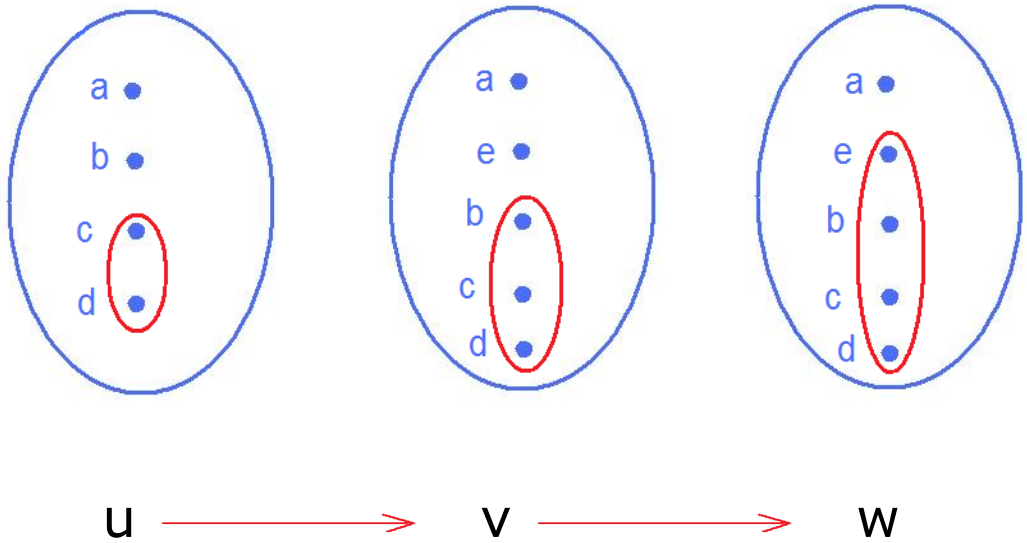
A Kripke sheaf over F is a triple $\mathscr{F}= (F,D,\rho)$, for which (F^*,D,ρ) is a Kripke sheaf over F^*.

Kripke sheaf models and forcing are defined in a straightforward way:

$M,u \vDash a=b$ iff a equals b

$M,u \vDash \Box B(d_1,\ldots, d_n)$ iff

for any v, uRv implies $M,v \vDash B(\rho_{uv}(d_1),\ldots,\rho_{uv}(d_n))$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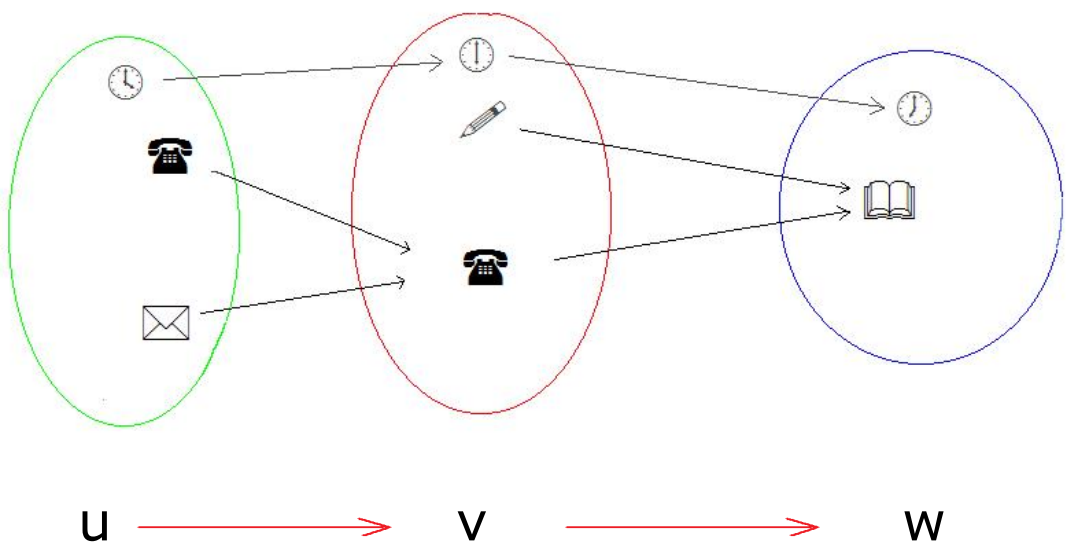
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Theorem (equivalence of KFEs and Kripke sheaves)

- For any KFE Φ there exists a Kripke sheaf $\mathcal{F}$ such that

  $ML^{=}(\Phi) = ML^{=}(\mathcal{F})$

- For any Kripke sheaf $\mathcal{F}$ there exists a KFE Φ such that

  $ML^{=}(\Phi) = ML^{=}(\mathcal{F})$

- Similarly for the intuitionistic case.

We need these generalizations of Kripke frame semantics, because the basic logic with equality **QK**$^{=}$ is Kripke-incomplete. In fact, take the formula

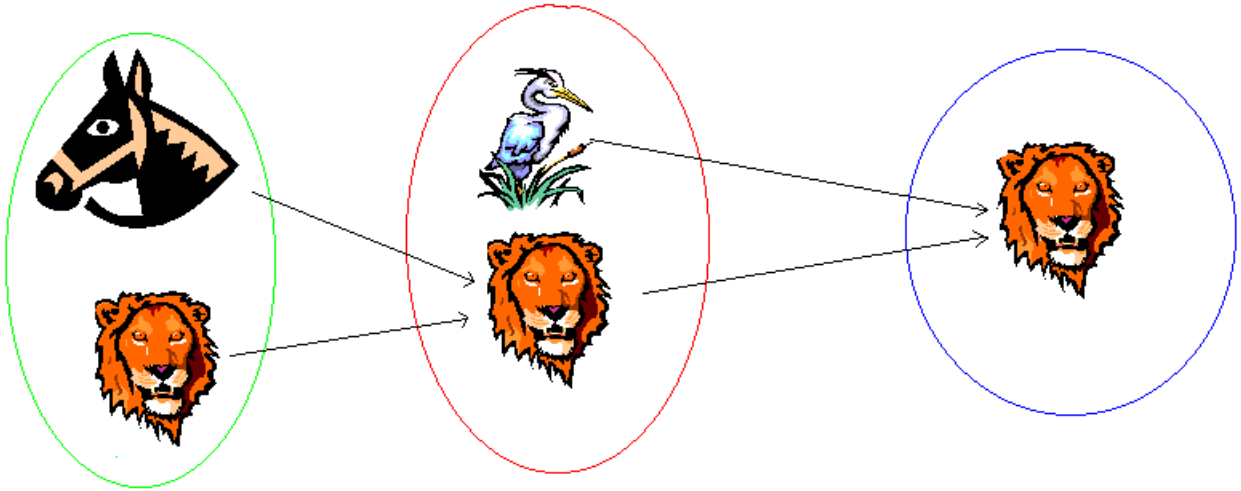
$$\mathrm{CE} := \forall x\, \forall y\, (x \neq y \rightarrow \Box\, (x \neq y))$$

It is true in every usual Kripke model, but does not belong to **QK**$^{=}$ (see below).

The same happens in the intuitionistic case. The axiom of *decidable equality*

$$\mathrm{DE} := \forall x\, \forall y\, (x = y \vee x \neq y)$$

is true in every intuitionistic Kripke model, but does not belong to **QH**$^{=}$.



$$u \longrightarrow v \longrightarrow w$$

This Kripke sheaf refutes CE;

its transitive closure refutes DE.

<u>Def</u> The logic (of a certain type) of a class of frames $\mathscr{C}$ is the intersection of the logics of frames from $\mathscr{C}$.

A logic of a class of Kripke frames is called Kripke ($\mathscr{K}$)-complete.

A logic of a class of Kripke sheaves (or KFEs) is called $\mathscr{KE}$-complete.

So the logics $\mathbf{QH}^{=}$ (and $\mathbf{QL}^{=}$ for any nonclassical intermediate L), $\mathbf{QK}^{=}$ (and $\mathbf{QL}^{=}$ for any nontrivial modal **L**) are $\mathscr{K}$-incomplete.

Problem

How to restore completeness? Is it true that:

- L is a $\mathscr{K}$-complete mpl $\Rightarrow$ $L^{=}$+CE is a $\mathscr{K}$-complete mpl=
- L is a $\mathscr{K}$-complete spl $\Rightarrow$ $L^{=}$+DE is a $\mathscr{K}$-complete spl=

Examples of Kripke-completeness

1. Surprisingly, for logics of the form **QL** not so many examples are known:

- for standard logics L (classical results by Kripke, Gabbay, Cresswell et al.)

  modal **K, T, D, B, K4, S4, S5**

  T: reflexive frames

  D: serial frames

  K4: transitive frames

  B: symmetric frames

  intuitionistic logic **H**

- for other cases, with more sophisticated proofs

  S4.2 = S4 + $\Diamond\Box A \rightarrow \Box\Diamond A$ (Ghilardi) confluent frames

K4.3 = K4 + $\Box(\Box A \wedge A \rightarrow B) \vee \Box(\Box B \wedge B \rightarrow A)$

non-branching transitive

S4.3 = K4.3 + $\Box A \rightarrow A$

K4.3 + $\Box\Box A \rightarrow A$ density

LC = H + $(A \rightarrow B) \vee (B \rightarrow A)$ non-brachning

(Corsi, 1990s)

2. For other kinds of logics see our book, Ch.6.

Barcan formula

$$Ba := \Diamond\exists x A \rightarrow \exists x \Diamond A$$

This formula is valid in a Kripke frame iff the domains *remain constant:*

if uRv then $D_u = D_v$

For he same basic cases, **QL**+Ba are also Kripke-complete

(but Ba is derivable in **QB, QS5**)

However, **QS4.2** + Ba is *K*-incomplete (SS 1990)

<u>Def</u> A propositional modal logic is called universal if the class of its frames is universal, i.e., the class of models of a universal classical first-order theory.

A propositional logic of a single finite frame is called tabular.

<u>Theorem</u> (Tanaka - Ono, 2001; book09) *If a modal propositional logic Λ is universal or tabular and K -complete, then L = QΛ +Ba is also K -complete.*

<u>Theorem</u> (Shimura 1993) The same holds for the intuitionistic case and L = QΛ +CD (the axiom of constant domains).

<u>Def</u> A modal predicate logic L is *strongly K* (or *KE*) *-complete* if every L-consistent theory Γ is satisfied at some Kripke frame (resp. KFE).

This means that Γ is true at some world in some Kripke model.

The same for the intuitionistic case: a theory is a pair of sets of formulas.

<u>Def</u> An mpl L is *conicallly expressive* if the master modality $\Box^*$ is expressible in L (eg if L is transitive).

<u>Theorems</u> (book 09) For any predicate logic L

(1) If L is strongly KE-complete, then $L^{=}$ is strongly KE-complete.

(2) If L is KE-complete, then $L^{=}$ is KE-complete

for any spl and conically expressive mpl

(3) $L^{=}$ is conservative over L

for any spl and conically expressive mpl

(4) L is [strongly] KE-complete iff $L^{=}$ is [strongly] KE-complete.

for any spl and conically expressive mpl.

(5) If L is K [KE]-complete then L+C is K [KE]-complete.

for any spl= and conically expressive mpl=

for any pure equality formula C (in particular for CE and DE).

We do not know about the converse to (5).

Note that $L^{=}$+C may be not conservative over L.

<u>Def</u> $L^{=c} := L^{=} + CE$ for an mpl L

$L^{=d} := L^{=} + DE$ for an spl L

<u>Theorem 1</u> (1) Suppose L is a *K*-complete mpl of one of the following types

- L is complete w.r.t. frames over trees,
- L contains $\Diamond\Box p \to \Box\Diamond p$,
- L contains Ba.

Then $L^{=c}$ is also *K*-complete.

(2) Suppose L is a *K*-complete spl of one of the following types

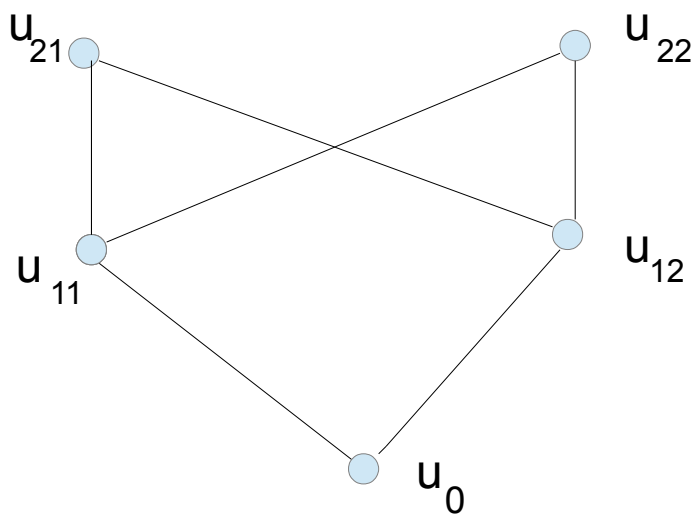
- L is complete w.r.t. frames over trees,
- L contains $J := \neg p \vee \neg\neg p$ (the *weak excluded middle*),
- L contains the *constant domains axiom*

  $CD := \forall x(P(x) \vee q) \to \forall x P(x) \vee q$.

Then $L^{=d}$ is also *K*-complete.

Note that conditions in theorem 1 are not necessary.

Counterexamples



Consider the class KF of Kripke frames over this propositional frame F.

<u>Theorem 2</u> Let L be an spl between $QH+J_2$ and **IL**(KF). Then $L^{=d}$ is *K*-incomplete.

$$J_2 := \neg(p \wedge q \wedge r) \rightarrow \neg(p \wedge q) \vee \neg(q \wedge r) \vee \neg(p \wedge r)$$

(the weak De Morgan law)

The idea of proof. There is a Kripke sheaf Φ over F and a formula A such that

- Φ validates DE
- Φ refutes A
- Every predicate Kripke frame over F validates A

Some open problems

1. Is $L^{=}$ conservative over L for any mpl L?
2. Is the Kripke-completion of $L^{=}$ always finitely axiomatizable over L?