The Frame of the p-Adic Numbers

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Outline

Pointfree Topology

What is pointfree topology?

It is an approach to topology based on the fact that the lattice of open sets of a topological space contains considerable information about the topological space.

"...what the pointfree formulation adds to the classical theory is a remarkable combination of elegance of statement, simplicity of proof, and increase of extent." R. Ball & J. Walters-Wayland.

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Motivation

The lattice of open subsets of X

Let X be a topological space and $\Omega(X)$ the family of all open subsets of X. Then $\Omega(X)$ is a complete lattice:

- $\bigvee U_i = \bigcup U_i$,
- $U \wedge V = U \cap V$,
- $\bigwedge U_i = int(\bigcap U_i)$,
- $1 = X$,
- $0 = \varnothing$.

Moreover,

$$U \wedge \bigvee V_i = \bigvee (U \wedge V_i).$$

Cont. I

Continous Functions

If $f : X \to Y$ is continuous, then $f^{-1} : \Omega(Y) \to \Omega(X)$ is a lattice homomorphism. Morover, it satisfies:

$$f^{-1}(\bigcup U_i) = \bigcup f^{-1}(U_i).$$

Frames and Frame Homomorphisms

Definition

A *frame* is a complete lattice L satisfying the distributivity law

$$\bigvee A \wedge b = \bigvee\{a \wedge b \mid a \in A\}$$

for any subset $A \subseteq L$ and any $b \in L$.

Let L and M be frames. A *frame homomorphism* is a map $h : L \to M$ satisfying

1. $h(0) = 0$ and $h(1) = 1$,
2. $h(a \wedge b) = h(a) \wedge h(b)$,
3. $h\left(\bigvee_{i \in J} a_i\right) = \bigvee \left\{h(a_i) : i \in J\right\}$.

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The category **Frm**

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Objects: Frames.
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A frame L is called *spatial* if it is isomorphic to $\Omega(X)$ for some X.

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The functor Ω

The contravariant functor Ω

$$\begin{aligned} \Omega : \mathbf{Top} &\to \mathbf{Frm} \\ X &\mapsto \Omega(X) \\ f &\mapsto \Omega(f), \text{ where } \Omega(f)(U) = f^{-1}(U). \end{aligned}$$

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A topological space X is *sober* if $\overline{\{x\}}^c$ are the only meet-irreducibles in $\Omega(X)$.

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Points in a frame

Motivation

The points x in a space X are in a one-one correspondence with the continuous mappings $f_x : \{*\} \to X$ given by $* \mapsto x$ and with the frame homomorphisms $f_x^{-1} : \Omega(X) \to \Omega(\{*\}) \cong \mathbf{2}$ whenever X is sober.

Definition

A point in a frame L is a frame homomorphism $h : L \to \mathbf{2}$.

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The functor Σ

The Spectrum of a Frame

Let L be a frame and for $a \in L$ set $\Sigma_a = \{h : L \to \mathbf{2} \mid h(a) = 1\}$.
The family $\{\Sigma_a \mid a \in L\}$ is a topology on the set of all frame homomorphisms $h : L \to \mathbf{2}$.
This topological space, denoted by ΣL, is the *spectrum* of L.

The functor Σ

$$\begin{aligned}\Sigma : \mathbf{Frm} &\to \mathbf{Top}\\ L &\mapsto \Sigma L\\ f &\mapsto \Sigma(f), \text{ where } \Sigma(f)(h) = h \circ f.\end{aligned}$$

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The Spectrum Adjunction

Theorem (see, e.g., *Frame and Locales*, Picado & Pultr [9])

The functors Ω and Σ form an adjoint pair.

Remark

The category of *sober spaces* and continuous functions is dually equivalent to the full subcategory of **Frm** consisting of *spatial frames*.

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Frame of $\mathbb{R}$

Definition (Joyal [6] and Banaschewski [1])

The *frame of the reals* is the frame $\mathcal{L}(\mathbb{R})$ generated by all ordered pairs (p, q), with $p, q \in \mathbb{Q}$, subject to the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$.

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$.

(R4) $1 = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$.

Remark

Banaschewski studied this frame with a particular emphasis on the pointfree extension of the ring of continuous real functions and proved pointfree version of the Stone-Weierstrass Theorem.

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The p-adic numbers

p-Adic Valuation

Fix a prime number $p \in \mathbb{Z}$. For each $n \in \mathbb{Z} \setminus \{0\}$, let $\nu_p(n)$ be the unique positive integer satisfying $n = p^{\nu_p(n)} m$ with $p \nmid m$.
For $x = a/b \in \mathbb{Q} \setminus \{0\}$, we set $\nu_p(x) = \nu_p(a) - \nu_p(b)$.

p-Adic Absolute Value

For any $x \in \mathbb{Q}$, we define $|x|_p = p^{-\nu_p(x)}$ if $x \neq 0$ and we set $|0|_p = 0$.

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The field $\mathbb{Q}_p$

Facts

- $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.
- $\mathbb{Q}_p$ is locally compact, totally disconnected, 0-dimensional, and metrizable.

Moreover, the open balls $S_r\langle a\rangle := \{x \in \mathbb{Q}_p : |x-a|_p < r\}$ satisfy the following:

- $b \in S_r\langle a\rangle$ implies $S_r\langle a\rangle = S_r\langle b\rangle$.
- $S_r\langle a\rangle \cap S_s\langle a\rangle \neq \varnothing$ iff $S_r\langle a\rangle \subseteq S_s\langle b\rangle$ or $S_s\langle b\rangle \subseteq S_r\langle a\rangle$.
- $S_r\langle a\rangle$ is open and compact.
- Every ball is a disjoint union of open balls of any smaller radius.

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The frame of $\mathbb{Q}_p$

Definition

Let $\mathcal{L}(\mathbb{Q}_p)$ be the frame generated by the elements $B_r(a)$, with $a \in \mathbb{Q}$ and $r \in |\mathbb{Q}| := \{p^{-n},\ n \in \mathbb{Z}\}$, subject to the following relations:

(Q1) $B_s(b) \leq B_r(a)$ whenever $|a - b|_p < r$ and $s \leq r$.

(Q2) $B_r(a) \wedge B_s(b) = 0$ whenever $|a - b|_p \geq r$ and $s \leq r$.

(Q3) $1 = \bigvee \{B_r(a) : a \in \mathbb{Q},\ \ r \in |\mathbb{Q}|\}$.

(Q4) $B_r(a) = \bigvee \{B_s(b) : |a - b|_p < r,\ s < r,\ b \in \mathbb{Q}\}$.

Properties of $\mathcal{L}(\mathbb{Q}_p)$

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- $B_r(a) = B_r(b)$ whenever $|a-b|_p < r$.
- $|a-b|_p < r$ implies $B_s(b) \leq B_r(a)$ or $B_s(b) \geq B_r(a)$.
- $B_r(a) = \bigvee \left\{ B_{r/p}(a + xp^{n+1}) \mid x = 0, 1, \ldots, p-1 \right\}$.

Theorem

Let $B_r(a) \in \mathcal{L}(\mathbb{Q}_p)$ a generator. Then $B_r(a)$ is complemented (clopen) and $B_r(a)' = \bigvee \{ B_r(b) \mid |a-b|_p \geq r \}$.

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The spectrum of $\mathcal{L}(\mathbb{Q}_p)$

Definition

For each $x \in \mathbb{Q}$, let $\sigma(x)$ be the unique frame homomorphism $\sigma(x) : \mathcal{L}(\mathbb{Q}_p) \to \mathbf{2}$ satisfying

$$\sigma(x)(B_r(a)) = \begin{cases} 1 & \text{if } |a-x|_p < r \\ 0 & \text{otherwise.} \end{cases}$$

Cont. I

Lemma

For each $x \in \mathbb{Q}_p$, the function $\varphi(x) : \mathcal{L}(\mathbb{Q}_p) \to \mathbf{2}$, defined on generators by

$$\varphi(x)(B_r(a)) = \lim_{n\to\infty} \sigma(x_n)(B_r(a)),$$

where $\{x_n\}$ is any sequence of rationals satisfying $\lim_{n\to\infty} x_n = x$, extends to a frame homomorphism on $\mathcal{L}(\mathbb{Q}_p)$ (viewing $\mathbf{2}$ as a discrete space).

The spectrum of $\mathcal{L}(\mathbb{Q}_p)$ is homeomorphic to $\mathbb{Q}_p$

Theorem

The function $\varphi : \mathbb{Q}_p \to \Sigma\mathcal{L}(\mathbb{Q}_p)$ defined by $x \mapsto \varphi(x)$ is a homeomorphism.

Corollary

The frame homomorphism $h : \mathcal{L}(\mathbb{Q}_p) \to \Omega(\mathbb{Q}_p)$ defined by $B_r(a) \mapsto S_r\langle a\rangle$ is an isomorphism.

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Continuous p-Adic Functions on a frame L

From the Adjunction between **Frm** and **Top**

For a topological space X, we get a bijection

$$\mathbf{Top}(X, \mathbb{Q}_p) \cong \mathbf{Frm}(\mathcal{L}(\mathbb{Q}_p), \Omega(X)).$$

This provides a natural extension of the classical notion of a continuous p-adic function.

Definition

A *continuous p-adic function* on a frame L is a frame homomorphism $\mathcal{L}(\mathbb{Q}_p) \to L$. We denote the set of all continuous p-adic functions on L by $\mathcal{C}_p(L)$.

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Cont. I

Example

Let $\lambda \in \mathbb{Q}_p$ and consider the function $f_\lambda : X \to \mathbb{Q}_p$ defined by $f_\lambda(x) = \lambda$ for all $x \in X$. Then $f_\lambda \in \mathcal{C}(X, \mathbb{Q}_p)$ and $f_\lambda^{-1} : \Omega(\mathbb{Q}_p) \to \Omega(X)$ is a frame homomorphism.
For any $a \in \mathbb{Q}$ and $r \in |\mathbb{Q}|$, we have

$$f_\lambda^{-1}(S_r\langle a\rangle) = \begin{cases} \mathbb{Q}_p & \text{if } |\lambda - a|_p < r, \\ \varnothing & \text{if } |\lambda - a|_p \geq r. \end{cases}$$

Cont. I

Example

For any frame L and $\lambda \in \mathbb{Q}_p$, the map $\boldsymbol{\lambda} : \mathcal{L}(\mathbb{Q}_p) \to L$ defined on the generators by

$$\boldsymbol{\lambda}(B_r(a)) = \begin{cases} 1 & \text{if } |\lambda - a|_p < r, \\ 0 & \text{if } |\lambda - a|_p \geq r. \end{cases}$$

is a frame homomorphism.

Operations in $\mathcal{C}_p(L)$

Example

For $f, g \in \mathcal{C}(X, \mathbb{Q}_p)$, $a \in \mathbb{Q}$, $r \in |\mathbb{Q}|$,

$$(f+g)^{-1}(S_r\langle b\rangle) = \bigcup_{q\in\mathbb{Q}} \{f^{-1}(S_r\langle q\rangle) \cap g^{-1}(S_r\langle a-q\rangle)\}$$

Cont. I

Definition

For $f, g \in \mathcal{C}_p(L)$, $a \in \mathbb{Q}$, $r \in |\mathbb{Q}|$, we define

$$(f+g)(B_r(a)) =$$
$$= \bigvee \left\{ f(B_{s_1}(b_1)) \wedge g(B_{s_2}(b_2)) \,|\, B_{s_1}\langle b_1\rangle + B_{s_2}\langle b_2\rangle \subseteq B_r\langle a\rangle \right\}$$
where $B_{s_1}\langle b_1\rangle + B_{s_2}\langle b_2\rangle = \left\{ x+y \,|\, x \in B_{s_1}\langle b_1\rangle,\ y \in B_{s_2}\langle b_2\rangle \right\}$,

and

$$(f \cdot g)(B_r(a)) =$$
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$\mathcal{C}_p(L)$ is a $\mathbb{Q}_p$-algebra

Theorem

For any frame L, $(\mathcal{C}_p(L), +, \cdot)$, with the above operations, is a commutative ring with unity.

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Idempotents in $\mathcal{C}_p(L)$

Motivation

The idempotents of $\mathcal{C}(X, \mathbb{Q}_p)$ are exactly the $\mathbb{Q}_p$-characteristic functions of clopen subsets of X.

Idempotents in $\mathcal{C}(X, \mathbb{Q}_p)$

Let $U \subseteq X$ be clopen and $\phi_U : X \to \mathbb{Q}_p$ be defined by $\phi_U(x) = 1$ if $x \in U$, and $\phi_U(x) = 0$ otherwise. Then

$$\phi_u^{-1}(S_r\langle a\rangle) = \begin{cases} \mathbb{Q}_p & \text{if } 0 \in S_r\langle a\rangle \text{ and } 1 \in S_r\langle a\rangle \\ U & \text{if } 0 \notin S_r\langle a\rangle \text{ and } 1 \in S_r\langle a\rangle \\ U^c & \text{if } 0 \in S_r\langle a\rangle \text{ and } 1 \notin S_r\langle a\rangle \\ \varnothing & \text{otherwise.} \end{cases}$$

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Theorem

Let L be a frame and let $u \in L$ be clopen. Then the function $\chi_u : \mathcal{L}(\mathbb{Q}_p) \to L$ defined on generators by

$$\chi_u(B_r(a)) = \begin{cases} 1 & \text{if } |a|_p < r \text{ and } |1-a|_p < r, \\ u & \text{if } |a|_p \geq r \text{ and } |1-a|_p < r, \\ u' & \text{if } |a|_p < r \text{ and } |1-a|_p \geq r, \\ 0 & \text{otherwise.} \end{cases}$$

is a frame homomorphism.

Cont. II

Theorem

Let L be a frame. Then $f \in \mathcal{C}_p(L)$ is an idempotent if and only if $f = \chi_u$ for some clopen element $u \in L$.

A norm in $\mathcal{C}_p(L)$

Motivation

If X is compact Hausdorff, then $||f|| = \sup\{|f(x)|_p\}$ is a norm. Note that

$$||f|| = p^{-n} \iff f(x) \in S_{p^{-n+1}}\langle 0\rangle \text{ for all } x \in X$$
$$\iff f^{-1}(S_{p^{-n+1}}\langle 0\rangle) = X.$$

Theorem

Let L be a compact regular frame. For each $h \in \mathcal{C}_p(L)$, define

$$||h|| = \inf\{p^{-n} : n \in \mathbb{Z},\, h(B_{p^{-n+1}}(0)) = 1\}.$$

Then, $||\cdot||$ is a norm on $\mathcal{C}_p(L)$.

A norm in $\mathcal{C}_p(L)$

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About the Stone-Weierstrass Theorem

Dieudonné [2] (1944)

The ring $\mathbb{Q}_p[X]$ of polynomials with coefficients in $\mathbb{Q}_p$ is dense in the ring $\mathcal{C}(F, \mathbb{Q}_p)$ of continuous functions on a compact subset F of $\mathbb{Q}_p$ with values in $\mathbb{Q}_p$.

Kaplansky [7] (1950)

If $\mathbb{F}$ is a nonarchimedean valued field and X is a compact Hausdorff space, then any unitary subalgebra $\mathcal{A}$ of $\mathcal{C}(X, \mathbb{F})$ which separates points is uniformly dense in $\mathcal{C}(X, \mathbb{F})$.

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Point-Separation

Definition

Let $\mathbb{F}$ be a field. A unitary subalgebra $\mathcal{A} \in \mathcal{C}(X, \mathbb{F})$ is said to *separate points* if, for any pair of distinct points x and y, there is a function f_α such that $f_\alpha(x) = 0$ and $f_\alpha(y) = 1$.

Theorem (Kaplansky [7] and [10])

Let X be a compact Hausdorff (0-dimensional) space and let $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{Q}_p)$ be a unitary subalgebra. Then $\mathcal{A}$ separates points iff for any clopen subset $U \subseteq X$, the $\mathbb{Q}_p$-characteristic function ϕ_U belongs to the closure of $\mathcal{A}$ in $\mathcal{C}(X, \mathbb{Q}_p)$.

Point-Separation

Definition

Let $\mathbb{F}$ be a field. A unitary subalgebra $\mathcal{A} \in \mathcal{C}(X, \mathbb{F})$ is said to *separate points* if, for any pair of distinct points x and y, there is a function f_α such that $f_\alpha(x) = 0$ and $f_\alpha(y) = 1$.

Theorem (Kaplansky [7] and [10])

Let X be a compact Hausdorff (0-dimensional) space and let $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{Q}_p)$ be a unitary subalgebra. Then $\mathcal{A}$ separates points iff for any clopen subset $U \subseteq X$, the $\mathbb{Q}_p$-characteristic function ϕ_U belongs to the closure of $\mathcal{A}$ in $\mathcal{C}(X, \mathbb{Q}_p)$.

Point-Separation Pointfree

Remark

If L is a compact regular frame, then it is spatial and

$$\mathbf{Top}(\Sigma L, \mathbb{Q}_p) \cong \mathbf{Frm}(\mathcal{L}(\mathbb{Q}_p), L).$$

Definition

Let L be a compact 0-dimensional frame. We say that a unitary subalgebra $\mathcal{A}$ of $\mathcal{C}_p(L)$ separates points if its closure contains the idempotents of $\mathcal{C}_p(L)$.

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Stone-Weierstrass Theorem in Pointfree Topology

Theorem

Let L be a compact 0-dimensional frame and let $\mathcal{A}$ be a unitary subalgebra of $\mathcal{C}_p(L)$ which separates points. Then $\mathcal{A}$ is uniformly dense in $\mathcal{C}_p(L)$.

Thank you!

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