Bunched hypersequent calculi for distributive substructural logics and extensions of BI

Agata Ciabattoni and Revantha Ramanayake

Technische Universität Wien, Austria

TACL 2017 (Prague)

26-30 June 2017
The Lambek calculus with exchange FL_e

\[ p \Rightarrow p \quad \perp, \Gamma \Rightarrow D \quad \Rightarrow 1 \quad \Gamma \Rightarrow \top \quad \frac{A_1, A_2, \Gamma \Rightarrow D}{A_1 \otimes A_2, \Gamma \Rightarrow D} \quad \otimes l \]

\[ \frac{\Gamma \Rightarrow A, \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B} \quad \otimes r \]

\[ \frac{\Gamma \Rightarrow A, B, \Delta \Rightarrow D}{A \rightarrow B, \Gamma, \Delta \Rightarrow D} \quad \rightarrow l \]

\[ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \quad \rightarrow r \]

\[ \frac{A_i, \Gamma \Rightarrow D}{A_1 \land A_2, \Gamma \Rightarrow D} \quad \land l \]

\[ \frac{\Gamma \Rightarrow A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \land B} \quad \land r \]

\[ \frac{A, \Gamma \Rightarrow D, B, \Gamma \Rightarrow D}{A \lor B, \Gamma \Rightarrow D} \quad \lor l \]

\[ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \lor A_2} \quad \lor r \]

- The antecedent is a comma-separated list of formulae
- Cut-elimination holds so we omit the cut rule.
- This calculus has the **subformula** property: every formula occurring in the premise of a rule instance is a subformula of some formula in the conclusion.
- The subformula property is the **point**. It restricts the formulae that can appear in a sequent/derivation to subformulae of the endsequent \( \Rightarrow A \).
- We can use a calculus with the subformula property to give decidability, complexity, proof search, interpolation, standard completeness arguments...
An observation: \( \text{FL}_e \) is not distributive

\[ A \land (B \lor C) \Rightarrow (A \land B) \lor (A \land C) \] is not derivable

Proof: what rule can be applied to obtain this sequent? (4 possibilities)

\[
\begin{align*}
A &\Rightarrow (A \land B) \lor (A \land C) \\
\hline
A \land (B \lor C) &\Rightarrow (A \land B) \lor (A \land C) \\
\end{align*}
\]
\( \land l \)

\[
\begin{align*}
B \lor C &\Rightarrow (A \land B) \lor (A \land C) \\
\hline
A \land (B \lor C) &\Rightarrow (A \land B) \lor (A \land C) \\
\end{align*}
\]
\( \land l \)

\[
\begin{align*}
A \land (B \lor C) &\Rightarrow A \land B \\
\hline
A \land (B \lor C) &\Rightarrow (A \land B) \lor (A \land C) \\
\end{align*}
\]
\( \lor r \)

\[
\begin{align*}
A \land (B \lor C) &\Rightarrow A \land C \\
\hline
A \land (B \lor C) &\Rightarrow (A \land B) \lor (A \land C) \\
\end{align*}
\]
\( \lor r \)

By inspection, none of the premises is derivable.

Proof calculi for distributive substructural logics have been studied e.g. in the context of relevant logics.
A bunch calculus sDFL$_e$ for DFL$_e$ (Dunn 1974, Mints 1976)

- The antecedent has two structure connectives: comma and semicolon
- Comma $\rightsquigarrow$ multiplicative connectives. Semicolon $\rightsquigarrow$ additive connectives
Derivation of $A \land (B \lor C) \Rightarrow (A \land B) \lor (A \land C)$ in $sDFL_e$

$$
\begin{align*}
A & \Rightarrow A \\
A; B & \Rightarrow A \\
A; B & \Rightarrow A \land B \\
A; B \lor C & \Rightarrow (A \land B) \lor (A \land C) \\
A \land (B \lor C) & \Rightarrow (A \land B) \lor (A \land C)
\end{align*}
$$
Bunched (hyper)sequent calculi for distributive substructural logics

- How can we construct calculi with the subformula property for axiomatic extensions of DFL$_e$?

- (Ciabattoni, Galatos, Terui 2008) develop a general method for (hyper)sequent calculi

- To extend these methods to bunched (hyper)sequent calculi we
  
  1. Interpret the additional structure and prove a cut-elimination theorem on this extended structure.
  2. (This yields an algorithm for transforming an axiom into a structural rule)
  3. Characterise those axiom extensions that can be presented
  4. We also consider the special case of the logic of bunched implication BI (DFL$_e$ with two implications defined on $\Rightarrow$) where the above interpretation does not hold.

- Underlying aim: present logics in a simple extension of the sequent calculus, to permit applications of the calculus

  e.g. decidability, complexity, proof search, interpolation, standard completeness arguments
ASIDE: syntactic decidability arguments can be tricky. The (scom) rule.

\[ p \Rightarrow p \]
\[ \frac{A, B, \Gamma \Rightarrow D}{\therefore A \otimes B, \Gamma \Rightarrow D} \quad (\otimes l) \]
\[ \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\therefore \Gamma, \Delta \Rightarrow A \otimes B} \quad (\otimes r) \]

\[ \frac{\Gamma, A, B, \Delta \Rightarrow D}{\therefore \Gamma, B, A, \Delta \Rightarrow D} \quad (e) \]
\[ \frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow D}{\therefore A \rightarrow B, \Gamma, \Delta \Rightarrow D} \quad (\rightarrow l) \]
\[ \frac{A_{1}^{\alpha_{1}}, A_{2}^{\alpha_{2}}, \ldots, A_{n}^{\alpha_{n}} \Rightarrow D}{\therefore A_{1}^{2\alpha_{1}-r_{1}}, A_{2}^{2\alpha_{2}-r_{2}}, \ldots, A_{n}^{2\alpha_{n}-r_{n}} \Rightarrow D} \quad (\text{scom}) \]
\[ \frac{A_{1}^{\alpha_{1}}, A_{2}^{\alpha_{2}}, \ldots, A_{n}^{\alpha_{n}} \Rightarrow D}{\therefore \Gamma \Rightarrow D} \quad (\text{weak}) \]

\[ \frac{\Gamma \Rightarrow D}{\therefore \Gamma, \Delta \Rightarrow D} \]

▶ Is sMILL+(scom)+(weak) decidable? Is sMILL+(scom) decidable?

▶ The (scom) rule increases sequent size. To ensure termination of backward proof search, we note that only finitely many (scom) rules are required.

▶ In sMILL+(scom)+(weak), we can bound the number of (scom) rules applied to \( A_{1}^{\alpha_{1}} A_{2}^{\alpha_{2}} \Rightarrow D \) by omitting redundant derivations i.e. it is safe to omit derivations with repeated nodes along a branch up to \( \leq \) etc.

Def: \((X \Rightarrow D) \leq (Y \Rightarrow D)\) if latter can be derived from former by weakening

▶ However... bounding \( A_{1}^{\alpha_{1}} A_{2}^{\alpha_{2}} A_{3}^{\alpha_{3}} \Rightarrow D \) is problematic...
Example due to Nick Galatos.
Back to main topic: computing a rule from an axiom:
A calculus for $\text{DFL}_e + (1 \land (p \otimes q)) \to p$ (the axiom of restricted weakening)

- Use invertible rules backwards on axiom:

$$
\begin{align*}
1, (1; (p, q)) &\Rightarrow p \\
1, (1 \land (p \otimes q)) &\Rightarrow p \\
1 &\Rightarrow (1 \land (p \otimes q)) \to p
\end{align*}
$$

- So it suffices to derive $1, (1; (p, q)) \Rightarrow p$. In the presence of cut the following equivalences hold (‘Ackermann’s lemma’)

1) $1, (1; (p, q)) \Rightarrow p$

2) $X \Rightarrow p$

$$
\begin{align*}
\varnothing_a, (\varnothing_a; (X, q)) &\Rightarrow p
\end{align*}
$$

3) $X \Rightarrow p$

$$
\begin{align*}
Y &\Rightarrow q \\
\varnothing_a, (\varnothing_a; (X, Y)) &\Rightarrow p
\end{align*}
$$

4) $X \Rightarrow p$

$$
\begin{align*}
Y &\Rightarrow q \\
\Gamma[p] &\Rightarrow B \\
\varnothing_a, (\varnothing_a; (X, Y)) &\Rightarrow B
\end{align*}
$$

- Apply all cuts to the premises (assuming termination) to get equivalent rules

$$
\begin{align*}
\Gamma[X] &\Rightarrow B \\
\varnothing_a, (\varnothing_a; (X, Y)) &\Rightarrow B
\end{align*}
$$

- $s\text{DFL}_e + r + \text{cut}$ is sound and complete for $\text{DFL}_e + (1 \land (p \otimes q)) \to p$. By our cut-elimination theorem: so is $s\text{DFL}_e + r$ and it has the subformula property.
An example where the argument fails

$$\text{DFL}_e + (p \rightarrow 0) \lor ((p \rightarrow 0) \rightarrow 0)$$

- Applying invertible rules to $1 \Rightarrow (p \rightarrow 0) \lor ((p \rightarrow 0) \rightarrow 0)$ we get

  $$\varnothing \Rightarrow (p \rightarrow 0) \lor ((p \rightarrow 0) \rightarrow 0)$$

- Applying Ackermann lemma (below left), then invertible rule ($\lor$):

  $$\begin{array}{c}
  (p \rightarrow 0) \lor ((p \rightarrow 0) \rightarrow 0) \Rightarrow X \\
  \varnothing \Rightarrow X
  
  (p \rightarrow 0) \Rightarrow X \\
  ((p \rightarrow 0) \rightarrow 0) \Rightarrow X \\
  \varnothing \Rightarrow X
  
  \end{array}$$

- The rule above right violates the subformula property...

- ...and yet there is no way to proceed. There are no invertible rules to apply.

- And Ackermann’s lemma does not simplify premises

- It seems that structural rules extensions of $s\text{DFL}_e$ are not expressive enough to present $\text{DFL}_e + (p \rightarrow 0) \lor ((p \rightarrow 0) \rightarrow 0)$

- We need to extend the sequent formalism further...
Bunched hypersequent calculus for $\text{DFL}_e + (p \rightarrow 0) \lor ((p \rightarrow 0) \rightarrow 0)$ \hspace{1cm} (I)

- A natural extension of a sequent $\Gamma \Rightarrow A$ is to a non-empty set of sequents (Avron 1996, Pottingern 1983)

\[ \Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2 \mid \ldots \mid \Gamma_{n+1} \Rightarrow A_{n+1} \]

- Here we take the analogous extension of $\text{sDFL}_e$ with hypersquent structure
- The hypersequent calculus $\text{hDFL}_e$ is obtained from $\text{sDFL}_e$ as follows:

Add a hypersequent context “$g \mid$" to each rule. Also add rules manipulating the components

\[
\begin{align*}
\frac{g \mid \Gamma, A \Rightarrow B}{g \mid \Gamma \Rightarrow A \rightarrow B} \quad \rightarrow_r \\
\frac{h \mid h \mid g}{h \mid g} \quad \text{EC} \\
\frac{g}{h \mid g} \quad \text{EC}
\end{align*}
\]
Bunched hypersequent calculus for $\text{DFL}_e + (p \to 0) \lor ((p \to 0) \to 0)$ (II)

- Prove soundness of $h\text{DFL}_e$ wrt $\text{DFL}_e$ interpreting $|$ as disjunction
- Contrast with hypersequent calculi for extensions of FL$_e$ where
  $\Gamma_1 \Rightarrow A_1 \mid \Gamma_2 \Rightarrow A_2 \rightsquigarrow ((\Gamma'_1 \to A_1) \land 1) \lor ((\Gamma'_2 \to A_2) \land 1))$
- Therefore the following is an equivalent calculus.

$$h\text{DFL}_e + g \mid 1 \Rightarrow p \to 0 \mid 1 \Rightarrow (p \to 0) \to 0$$

- Applying invertible rules:

$$g \mid 1 \Rightarrow p \to 0 \mid 1 \Rightarrow (p \to 0) \to 0 \quad g \mid \emptyset_m, p \Rightarrow O_m \mid \emptyset_m, p \to 0 \Rightarrow O_m$$

- Now repeatedly apply Ackermann’s lemma to above right to get:

$$g \mid X \Rightarrow p \
g \mid Y \Rightarrow p \to 0$$

$$g \mid \emptyset_m, X \Rightarrow O_m \mid \emptyset_m, Y \Rightarrow O_m$$

- Applying invertible rules and all possible cuts we obtain a structural rule

$$g \mid X \Rightarrow p \
g \mid p, Y \Rightarrow O_m$$

$$g \mid \emptyset_m, X \Rightarrow O_m \mid \emptyset_m, Y \Rightarrow O_m$$

$$g \mid X, Y \Rightarrow O_m$$

$r$

- $h\text{DFL}_e + r$ (via cut-elimination) calculus for $\text{DFL}_e + (p \to 0) \lor ((p \to 0) \to 0)$
The substructural hierarchy over DFL$_e$

- We can characterise the extensions of DFL$_e$ that can be presented
- Following (Ciabattoni, Galatos, Terui 2008), set $N^d_0$ and $P^d_0$ as the set of propositional variables. Then define:

$$P^d_{n+1} ::= 1 \mid N^d_n \mid P^d_{n+1} \otimes P^d_{n+1} \mid P^d_{n+1} \land P^d_{n+1} \mid P^d_{n+1} \lor P^d_{n+1}$$

$$N^d_{n+1} ::= O_m \mid P^d_n \mid N^d_{n+1} \land N^d_{n+1} \mid P^d_{n+1} \to N^d_{n+1}$$

- The positive classes $P_i$ contain formulae whose most external connective is invertible on the left
- The negative classes $N_i$ contain formulae whose most external connective is invertible on the right

**Theorem**

*Every extension of DFL$_e$ by a disjunction of $N^d_2$ axioms computes a structural rule extension of hDFL$_e$ when the cuts on the premises terminate.*
The logic of bunched implications BI (O’Hearn and Pym, 1999)

- BI can be used for resource composition and systems modelling and as a propositional fragment of separation logic
- The calculus has a multiplicative implication $\rightarrow^*$ and an intuitionistic implication $\rightarrow$:

$$
\frac{\Gamma \Rightarrow A \quad \Sigma[B] \Rightarrow D}{\Sigma[\Gamma; A \rightarrow B] \Rightarrow D} \quad \rightarrow^* \quad \frac{A; \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \rightarrow^r
$$

- Algebraic semantics: Heyting (intuitionistic) algebras with a commutative monoidal operation $\otimes$ and residuated implication $\rightarrow^*$

  i.e. $x \otimes y \leq z$ iff $x \leq y \rightarrow^* z$ where $\leq$ is the Heyting partial order

- In other words: commutative bounded GBI-algebras (Galatos and Jipsen)
A calculus for $Bl + 1 \Rightarrow p \lor (p \rightarrow \bot)$ (BBI): an attempt (I)

- Boolean BI: BI with intuitionistic logic replaced by classical logic
- BBI is the propositional basis of separation logic (more widely used than BI)
- BBI is undecidable
- We cannot extend BI by permitting multiple formulae in the succedent (analogous of LJ $\iff$ LK) because cut-elimination fails due to the two types of structural connectives in the antecedent
- Idea: add hypersequent structure to sBI to interpret as before:
  
  $1 \Rightarrow p \lor (p \rightarrow \bot)$  
  $1 \Rightarrow p | 1 \Rightarrow (p \rightarrow \bot)$

- However: because we now have two right implication rules, the (logical) interpretation of $\Rightarrow$ in not clear
- If we cannot interpret $\Rightarrow$ then we cannot interpret $|$ 
- This means that we cannot obtain a calculus as we did before
A calculus for $Bl + 1 \Rightarrow p \lor (p \rightarrow \bot)$ (BBI): an attempt (II)

- Nevertheless we can consider the sequent consequences of the hypersequent calculus $hBl + r$ for some structural rule $r$

$$\{ \Gamma \Rightarrow A \mid \Gamma \Rightarrow A \text{ derivable in } hDFL_e + r \}$$

- Our proof of cut-elimination extends to structural rule extensions of $hBl$
- Idea: add a structural rule which derives desired sequent, use the subformula property to check the consistency of structural rule extensions
- It remains to interpret wrt the semantics of BI (future work)
- Aside. Recent work (Ciabattoni, Galatos, Terui 2016) interprets for (non-commutative) FL as a special disjunction built from ‘interated conjugates’
- Can we find interesting resource interpretations for such logics? Can we regain decidability for BBI-like logics?