Yes/No Formulae as a Description of Theories of Intuitionistic Kripke Models

Małgorzata Kruszelnicka

Institute of Mathematics
University of Opole
Opole, Poland

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Classical Model Theory

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- Ehrenfeucht–Fraïssé Games
- Fraïssé–Hintikka theorem

Kripke Model Theory

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Find a description for the notion of logical equivalence of two Kripke models by means of a single formula.

Model Description for IPC

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$$N_{\alpha,n}(\vec{p}) = \bigvee\{\varphi \in I_n(\vec{p})\colon \alpha \nVdash \varphi\}.$$

Kripke model

By a Kripke model for a first-order language L we mean a structure $\mathcal{K} = (K, \leq, \{K_\alpha : \alpha \in K\}, \Vdash)$. To any node $\alpha \in K$ there is assigned a classical first-order structure K_α for L.

For any two nodes $\alpha, \alpha' \in K$ we require that

$$\alpha \leq \alpha' \Rightarrow K_\alpha \subseteq K_{\alpha'}.$$

Forcing relation $\Vdash_{\mathcal{K}}$

Consider a node $\alpha \in K$ and a sequence $\bar{a} := a_1, \ldots, a_n$ of elements of the structure K_α, we put

- $\alpha \nVdash_{\mathcal{K}} \bot$ and $\alpha \Vdash_{\mathcal{K}} \top$
- $\alpha \Vdash_{\mathcal{K}} \varphi[\bar{a}] \iff K_\alpha \models \varphi[\bar{a}]$ for all atomic formulas $\varphi(\bar{x})$
- $\alpha \Vdash_{\mathcal{K}} (\varphi \wedge \psi)[\bar{a}] \iff \alpha \Vdash_{\mathcal{K}} \varphi[\bar{a}]$ and $\alpha \Vdash_{\mathcal{K}} \psi[\bar{a}]$
- $\alpha \Vdash_{\mathcal{K}} (\varphi \vee \psi)[\bar{a}] \iff \alpha \Vdash_{\mathcal{K}} \varphi[\bar{a}]$ or $\alpha \Vdash_{\mathcal{K}} \psi[\bar{a}]$
- $\alpha \Vdash_{\mathcal{K}} (\varphi \to \psi)[\bar{a}] \iff \forall_{\alpha' \geq \alpha}(\alpha' \Vdash_{\mathcal{K}} \varphi[\bar{a}] \Rightarrow \alpha' \Vdash_{\mathcal{K}} \psi[\bar{a}])$
- $\alpha \Vdash_{\mathcal{K}} \exists_y \varphi[\bar{a}, y] \iff \alpha \Vdash_{\mathcal{K}} \varphi[\bar{a}, b]$ for some element $b \in K_\alpha$
- $\alpha \Vdash_{\mathcal{K}} \forall_y \varphi[\bar{a}, y] \iff \forall_{\alpha' \geq \alpha} \alpha' \Vdash_{\mathcal{K}} \varphi[\bar{a}, b]$ for all elements $b \in K_{\alpha'}$

Formula's characteristic

As a measure of formula's complexity, we define the *characteristic* of a formula $\varphi(\overline{x})$, $char(\varphi)$. We put $char(\varphi) := (^{\rightarrow}p, ^{\forall}q, ^{\exists}r)$ if and only if there are

- p nested implications in φ,
- q nested universal quantifiers in φ,
- r nested existential quantifiers in φ.

- We put $({}^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r) \preceq ({}^{\rightarrow}p', {}^{\forall}q', {}^{\exists}r')$ whenever (p, q, r) precedes (p', q', r') with respect to the product order.

Logical equivalence

Given two Kripke models $\mathcal{K} = (K, \leq, \{K_\alpha \colon \alpha \in K\}, \Vdash)$ and $\mathcal{M} = (M, \leq, \{M_\beta \colon \beta \in M\}, \Vdash)$, for nodes $\alpha \in K$, $\beta \in M$ and any sequences $\overline{a}$ and $\overline{b}$ of elements of structures K_α and M_β respectively, we define a relation $\equiv_{p,q,r}$ as follows

$$(\alpha, \overline{a}) \equiv_{p,q,r} (\beta, \overline{b}) :\Longleftrightarrow (\alpha \Vdash_{\mathcal{K}} \varphi[\overline{a}] \Leftrightarrow \beta \Vdash_{\mathcal{M}} \varphi[\overline{b}])$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \leq ({}^{\to}p, {}^{\forall}q, {}^{\exists}r)$.

Strongly finite Kripke model

We say that model $\mathcal{K}$ is *strongly finite* if and only if both the frame $(K, \leq)$ and first-order structures assigned to the nodes of K are finite.

Consider a strongly finite Kripke model $\mathcal{K} = (K, \leq, \{K_\alpha \colon \alpha \in K\}, \Vdash)$ and its node $\alpha \in K$. Let $\overline{a}$ be a sequence of elements of the structure K_α. With a symbol

$$Y^{\alpha,\overline{a}}_{p,q,r}$$

we will denote a formula of characteristic at most $(^{\rightarrow}p, ^{\forall}q, ^{\exists}r)$ that is forced at α by the tuple $\overline{a}$. Similarly, a formula of characteristic at most $(^{\rightarrow}p, ^{\forall}q, ^{\exists}r)$ that is refuted at α by the tuple $\overline{a}$ will be denoted by

$$N^{\alpha,\overline{a}}_{p,q,r}.$$

Formulas $Y^{\alpha,\bar{a}}_{p,q,r}$ and $N^{\alpha,\bar{a}}_{p,q,r}$ are defined inductively over $p, q, r \geq 0$ in the following way:

$$Y^{\alpha,\bar{a}}_{0,0,0}(\bar{x}) = \Big(\bigwedge\{\varphi\colon \mathit{char}(\varphi) = ({}^{\to}0, {}^{\forall}0, {}^{\exists}0), \alpha \Vdash \varphi(\bar{a})\}\Big)(\bar{x})$$

$$N^{\alpha,\bar{a}}_{0,0,0}(\bar{x}) = \Big(\bigvee\{\varphi\colon \mathit{char}(\varphi) = ({}^{\to}0, {}^{\forall}0, {}^{\exists}0), \alpha \nVdash \varphi(\bar{a})\}\Big)(\bar{x})$$

$$Y^{\alpha,\bar{a}}_{p+1,q,r}(\bar{x}) = \bigvee_{\alpha' \geq \alpha} (N^{\alpha',\bar{a}}_{p,q,r} \rightarrow Y^{\alpha',\bar{a}}_{p,q,r})(\bar{x})$$

$$N^{\alpha,\bar{a}}_{p+1,q,r}(\bar{x}) = \bigvee_{\alpha' \geq \alpha} (Y^{\alpha',\bar{a}}_{p,q,r} \rightarrow N^{\alpha',\bar{a}}_{p,q,r})(\bar{x})$$

$$Y^{\alpha,\bar{a}}_{p,q+1,r}(\overline{x}) = \forall_y \bigvee_{\alpha' \geq \alpha} \bigvee_{a \in K_{\alpha'}} Y^{\alpha',\bar{a}a}_{p,q,r}(\overline{x}, y)$$

$$N^{\alpha,\bar{a}}_{p,q+1,r}(\overline{x}) = \bigvee_{\alpha' \geq \alpha} \bigvee_{a \in K_{\alpha'}} \forall_y \; N^{\alpha',\bar{a}a}_{p,q,r}(\overline{x}, y)$$

$$Y^{\alpha,\bar{a}}_{p,q,r+1}(\bar{x}) = \bigwedge_{a \in K_\alpha} \exists_y \; Y^{\alpha,\bar{a}a}_{p,q,r}(\bar{x}, y)$$

$$N^{\alpha,\bar{a}}_{p,q,r+1}(\bar{x}) = \exists_y \bigwedge_{a \in K_\alpha} N^{\alpha,\bar{a}a}_{p,q,r}(\bar{x}, y)$$

Theorem

Consider a strongly finite Kripke model $\mathcal{K}$ and a node $\alpha \in K$. Let $\bar{a}$ be a sequence of elements of the structure K_α. Then

$$Y^{\alpha,\bar{a}}_{p,q,r} \vdash Th_{p,q,r}(\alpha,\bar{a}) \quad \text{and} \quad N^{\alpha,\bar{a}}_{p,q,r} \vdash \widetilde{Th}_{p,q,r}(\alpha,\bar{a}).$$

Theorem

Consider strongly finite Kripke models $\mathcal{K}$ and $\mathcal{M}$, and nodes $\alpha \in K$, $\beta \in M$. Let $\bar{a}$ and $\bar{b}$ be sequences of elements of worlds K_α and M_β respectively. For $p, q, r \geq 0$,

$$(\alpha, \bar{a}) \equiv_{p,q,r} (\beta, \bar{b})$$

if and only if

$$\beta \Vdash_{\mathcal{M}} Y^{\alpha,\bar{a}}_{p,q,r}(\bar{b}) \quad \text{and} \quad \beta \nVdash_{\mathcal{M}} N^{\alpha,\bar{a}}_{p,q,r}(\bar{b}).$$

Theorem

Let $(\mathcal{K}, \alpha)$ and $(\mathcal{M}, \beta)$ be strongly finite rooted Kripke models. Then, for $p, q, r \geq 0$

$$(\mathcal{K}, \alpha) \equiv_{p,q,r} (\mathcal{M}, \beta)$$

if and only if

$$\beta \Vdash_{\mathcal{M}} Y^{\alpha,\bar{a}}_{p,q,r}(\bar{b}) \quad \text{and} \quad \beta \nVdash_{\mathcal{M}} N^{\alpha,\bar{a}}_{p,q,r}(\bar{b})$$

for all sequences $\bar{a}$ of K_α and $\bar{b}$ of M_β.

Thank you for your attention.

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Theorem (Ehrenfeucht–Fraïssé)

Two classical structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent with respect to all sentences with quantifier complexity not grater than n, $\mathcal{A} \equiv_n \mathcal{B}$, whenever there exists a winning strategy for Duplicator in Ehrenfeucht–Fraïssé game of length n on $\mathcal{A}$ and $\mathcal{B}$.

Theorem (Fraïssé–Hintikka, [3])

Let L be a first-order language with finite signature. Then we can effectively find for each $k, n < \omega$ a finite set $\Theta_{n,k}$ of unnested formulas $\theta(x_1, \ldots, x_n)$ of quantifier rank at most k, such that

(a) *for every first-order structure $\mathcal{A}$ of L and each n-tuple $\overline{a} = (a_1, \ldots, a_n)$ of elements of $\mathcal{A}$, there is exactly one formula $\theta \in \Theta_{n,k}$ such that $\mathcal{A} \models \theta(\overline{a})$,*

(b) *for every pair of first-order structures $\mathcal{A}, \mathcal{B}$ of L, if $\overline{a}, \overline{b}$ are respectively n-tuples of elements of $\mathcal{A}$ and $\mathcal{B}$, then $(\mathcal{A}, \overline{a}) \equiv_k (\mathcal{B}, \overline{b})$ if and only if there is $\theta \in \Theta_{n,k}$ such that $\mathcal{A} \models \theta(\overline{a})$ and $\mathcal{B} \models \theta(\overline{b})$,*

(c) *for every unnested formula $\varphi(\overline{x})$ with n free variables $\overline{x}$ and quantifier rank at most k, we can effectively find a disjunction $\theta_1 \vee \ldots \vee \theta_m$ of formulas $\theta_i(\overline{x}) \in \Theta_{n,k}$ which is logically equivalent to φ.*