Diagrammatic Duality

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or equivalent **lattice pairings** $\chi_\alpha \colon \mathsf{Fi}(L, \leq_\times) \times \mathsf{Fi}(L, \geq_+) \to \mathbf{2}$.

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Esakia spaces are partially ordered Stone spaces where the downset of each clopen subset is clopen.
Here T is the two-element Heyting algebra or Esakia space.

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Loosely: Each $\mathfrak{C}$-algebra C is **equivalent** to a diagram $\gamma\colon V \to \mathfrak{A}$.

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$((x \times_2 y) \times_1 z) \times_2 (y \times_1 z) = (x \times_2 y) \times_1 z$ is satisfied.

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Superproduct $L_1 \bowtie L_2$ of lattices $(L_i, \vee_i, \wedge_i)$ is the product $L_1 \times L_2$ with $\wedge = (\wedge_1, \vee_2)\,,\ \vee = (\vee_1, \wedge_2)\,,\ \cdot = (\wedge_1, \wedge_2)\,,\ + = (\vee_1, \vee_2)$.

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So interlaced bilattices are diagrammatic relative to lattices,

with discrete diagram $V = \boxed{\bullet \qquad \bullet}$.

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So Nelson algebras are diagrammatic relative to Heyting algebras,

with $V = \boxed{\bullet \longrightarrow \bullet}$ and image $H \to H^{\alpha}$ in $\mathbf{Heyt}$.

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Then each τ-algebra A is equivalent to a diagram $\alpha \colon \Omega_\infty \to \mathbf{Set}$
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Th: Algebras of any given type are diagrammatic relative to $\mathbf{Set}$.

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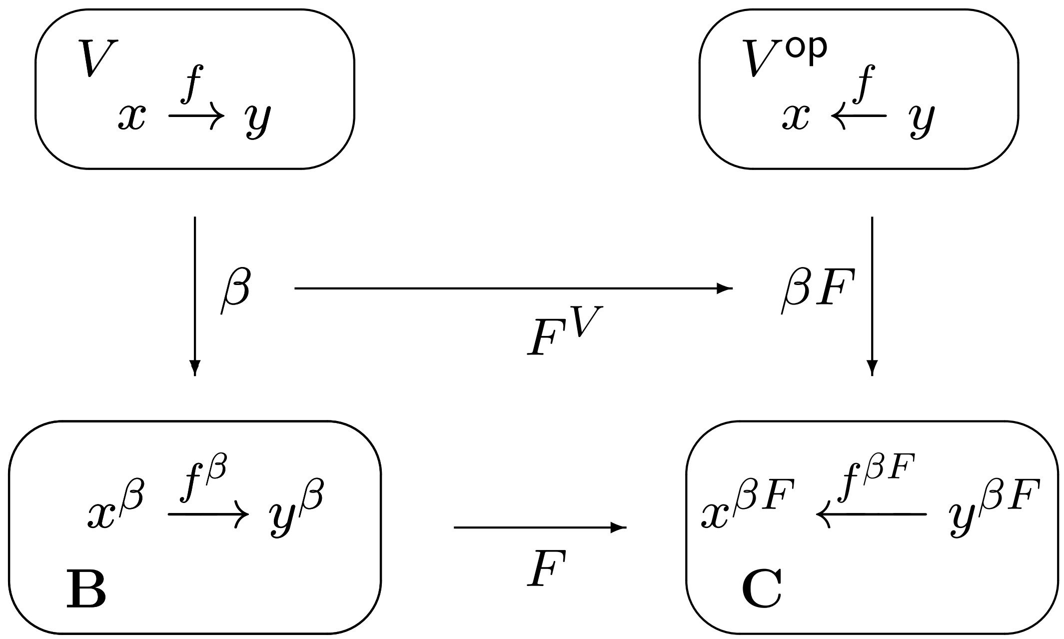

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restricts to a dual equivalence

$$\begin{tikzcd} \mathfrak{C}_{\mathfrak{A}} \arrow[r, bend left, "D^V"] & \mathfrak{X}_{\mathfrak{A}} \arrow[l, bend left, "E^V"] \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} \mathfrak{C} \arrow[r, bend left, "D^V"] & \mathfrak{X}_{\mathfrak{A}} \arrow[l, bend left, "E^V"] \end{tikzcd}$$

of $\mathfrak{C}$ with a subcategory $\mathfrak{X}_{\mathfrak{A}}$ of $\mathfrak{X}^{(V^{\mathsf{op}})}$.

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Residuated magma:

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Then CABAlistic duality gives labeled 3-**nets** as the dual spaces.

Thank you for your attention!