Arithmetic interpretation of the monadic fragment of intuitionistic predicate logic and Casari's formula

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joint work with Guram Bezhanishvili† and Kristina Brantley†

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TACL 2017, Prague, June 2017

Arithmetic interpretation of IPC

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Theorem (Grzegorczyk, Goldblatt, Boolos, Kuznetsov and Muravitsky)

For every formula φ of IPC,

$$\mathsf{IPC} \vdash \varphi \textit{ iff } \mathsf{Grz} \vdash \mathsf{t}(\varphi) \textit{ iff } \mathsf{GL} \vdash \mathsf{sp}(\mathsf{t}(\varphi)).$$

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The goal of this talk is to lift the above correspondences to the monadic setting as was anticipated by Esakia.

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- QGL and QGrz are not Kripke complete. (Montagna 1984, Ghilardi 1991)

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- QGL is not arithmetically complete. (Montagna 1984)
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Thus, arithmetic interpretation does not extend to the full predicate setting and a proof for the modal part of the correspondence would be essentially different than in the propositional case.

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- MGL is arithmetically complete. (Japaridze 1988)
- The (extended) Gödel–McKinsey–Tarski translation embeds MIPC into MGrz. (Fischer-Servi 1977)
- However, the (extended) splitting translation does not embed MGrz into MGL.

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$x \models \varphi \to \psi$ iff $y \models \varphi$ implies $x \models \psi$ for all $x \leq y$,

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$x \models \forall\varphi$	iff	$y \models \varphi$ for all $x \, Q \, y$, where $Q := \leq \circ E$.

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x \models \Box\varphi & \text{iff} & y \models \varphi \text{ for all } xRy, \\
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$$x \models \Box\varphi \quad \text{iff} \quad y \models \varphi \text{ for all } xRy,$$
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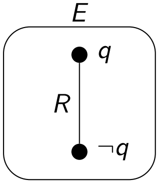

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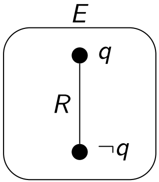

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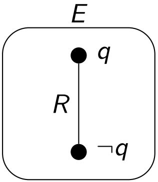

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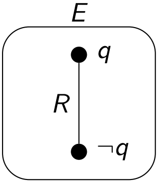

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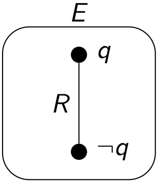

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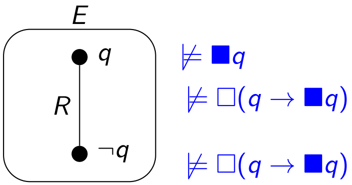

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- But MGL $\models \mathrm{sp}(\Psi)$.

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1. *A finite* MIPC*-frame validates* MCas *iff all its E-clusters are clean.*
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- Let $\mathrm{M^+IPC} = \mathrm{MIPC} + \mathrm{MCas}$ and let $\mathrm{M^+Grz} = \mathrm{MGrz} + \mathrm{t(MCas)}$.
- Note that $\mathrm{MGL} \vdash \mathrm{sp(t(MCas))}$, thus "$\mathrm{MGL} = \mathrm{M^+GL}$".

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- From a descriptive refutation frame (dual of a monadic Heyting algebra) we select a finite refutation frame.
- Note that descriptive $\mathsf{M}^{+}\mathsf{IPC}$-frames may have dirty clusters but the clusters in the maximum of an E-saturated clopen set are always clean.

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  Goal: $t \models \psi$ iff $\widehat{t} \models \psi$ for all $\psi \in \mathrm{Sub}(\varphi)$.
- For $t \in \mathfrak{G}$ consider the sets

$$
\begin{aligned}
\Sigma^{\exists}(t) &= \{\exists\delta \in \mathrm{Sub}(\varphi) : \widehat{t} \models \exists\delta\} \\
\Sigma^{\forall H}(t) &= \{\forall\beta \in \mathrm{Sub}(\varphi) : \widehat{t} \text{ is maximal wrt } \forall\beta\} \\
\Sigma^{\forall V}(t) &= \{\forall\gamma \in \mathrm{Sub}(\varphi) : \widehat{t} \not\models \forall\gamma \text{ but is not maximal wrt } \forall\gamma\} \\
\Sigma^{\rightarrow}(t) &= \{\alpha \rightarrow \sigma \in \mathrm{Sub}(\varphi) : \widehat{t} \not\models \alpha \rightarrow \sigma, \widehat{t} \not\models \alpha\}
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- We only add points t to $\mathfrak{G}$ if $\widehat{t}$ is from a clean cluster.
- Problematic case: Finding the right witnesses for formulas in $\Sigma^{\rightarrow}(t)$. Here we may introduce R-arrows in $\mathfrak{G}$ coming from original Q-arrows. (Here our proof differs from that of Grefe.)

Arithmetic interpretation of $\mathsf{M}^+\mathsf{IPC}$

Theorem

For every formula φ *of* MIPC,

$$\mathsf{M}^+\mathsf{IPC} \vdash \varphi \textit{ iff } \mathsf{M}^+\mathsf{Grz} \vdash \mathsf{t}(\varphi) \textit{ iff } \mathsf{MGL} \vdash \mathsf{sp}(\mathsf{t}(\varphi)).$$

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One variable fragments of predicate logics

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- For a formula φ of MIPC define a translation Ψ to QIPC by
  - $\Psi(p) = P(x)$, for each prop. letter p and a unary predicate $P(x)$,
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  - $\Psi(\forall\varphi) = \forall x \Psi(\varphi)$,
  - $\Psi(\exists\varphi) = \exists x \Psi(\varphi)$,
- Let MIPC $\subseteq$ L be an intuitionistic bi-modal logic and let QIPC $\subseteq$ S an intuitionistic predicate logic.

  L is the one-variable fragment of S iff for all φ of MIPC

$$L \vdash \varphi \text{ iff } S \vdash \Psi(\varphi)$$

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Corollary

M^+IPC *is the one-variable fragment of* Q^+IPC.

Thank you!