Quantifiers on languages and codensity monads

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Topological recognisers: BMs

A Boolean space with an internal monoid (BM, or BiM, for short) is a pair (X, M) where

- X is a Boolean space;
- M is a dense subspace of X equipped with a monoid structure;
- the biaction of M on itself extends to a biaction of M on X with continuous components.

(injectivity assumption, in the general framework, has to be dropped)

$$\begin{tikzcd} \beta(A^*) \arrow[r, "\tau"] & X \\ A^* \arrow[u, hook] \arrow[r] & M \arrow[u, hook] \end{tikzcd}$$

First-order quantifiers

Some quantifiers we are interested in:

- existential quantifier $\exists$;
- modular quantifiers $\exists_{p \bmod q}$. For $w \in (A \times 2)^*$, $w \vDash \exists_{p \bmod q} x.\psi(x)$ iff there exist exactly $p \bmod q$ positions in w for which the formula $\psi(x)$ is satisfied;
- semiring quantifiers $\exists_{k,S}$, for $(S, +, \cdot, 0_S, 1_S)$ a semiring and $k \in S$. If $w \in (A \times 2)^*$,

$$w \vDash \exists_{k,S} x.\psi(x) \;\Leftrightarrow\; \underbrace{1_S + \cdots + 1_S}_{m \text{ times}} = k,$$

where m is the number of positions in the word w that witness the validity of $\psi(x)$.

Question: Suppose (X, M) is a BM recognising the language $L_{\psi(x)}$. How to construct a BM recognising $L_{Qx.\psi(x)}$, for Q a certain (e.g. modular or semiring) quantifier?

[Gehrke-Petrișan-R 2016]: for $Q = \exists$, take $(\mathcal{V}X \times X, \mathcal{P}_f M \times M)$, where $\mathcal{V}X$ is the Vietoris space of X and $\mathcal{P}_f M$ is the finite powerset of M.

Hint for generalisation: $\mathcal{P}_f M$ is the free join-semilattice (=module over the two-element Boolean semiring) on M, and $\mathcal{V}X$ is the free profinite join-semilattice on X. In fact, $\mathcal{V}$ is the profinite monad of $\mathcal{P}_f$.

Codensity and profinite monads

The codensity monad (Kock 60s) of a functor $F\colon \mathbb{C} \to \mathbb{D}$ is the monad on $\mathbb{D}$ ‘best approximating the monad that F would induce if it had a left adjoint’.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, "F"'] \arrow[dr, phantom, "\sigma"] & \mathbb{D} \\
\mathbb{D} \arrow[ur, "K"'] \arrow[ur, bend right=60, "K'"'] \arrow[ur, phantom, bend right=30, "\varepsilon"'] &
\end{tikzcd}$$

$\forall \sigma' \colon K' \circ F \Rightarrow F\ \exists$ a unique $\varepsilon \colon K' \Rightarrow K$ s.t.

$$\sigma \circ \varepsilon F = \sigma'$$

The pair (K, σ) is called the codensity monad of F.
(Unit and multiplication of the monad by the universal property)

Codensity and profinite monads

If $\mathbb{C}$ is (essentially) small and $\mathbb{D}$ is complete, then $K\colon \mathbb{D} \to \mathbb{D}$ exists and is computed by $K(d) = \lim_{d \to F(c)} F(c)$.

Examples:

1. If $F\colon \mathsf{Set}_{\mathsf{fin}} \hookrightarrow \mathsf{Set}$, then $K = \beta\colon \mathsf{Set} \to \mathsf{Set}$.
2. If $F\colon \mathsf{sLat}_{\mathsf{fin}} \to \mathsf{BStone}$, then $K = \mathcal{V}\colon \mathsf{BStone} \to \mathsf{BStone}$.

If $\mathbb{V}$ is the category of algebras for a monad T on Set, the profinite monad of T is the codensity monad of $\mathbb{V}_{\mathsf{fin}} \to \mathsf{BStone}$ (cf. item 2).

We will be interested in monads T that model a FO quantifier.

Let $T\colon \mathsf{Set} \to \mathsf{Set}$ be a monad and $\widehat{T}\colon \mathsf{BStone} \to \mathsf{BStone}$ its profinite monad. Write $\mathbb{V}$ for the variety of T-algebras.

Lemma

For every Boolean space X, the following hold:

1. *$T|X|$ is dense in $\widehat{T}X$;*
2. *$\widehat{T}X$ is a profinite $\mathbb{V}$-algebra;*
3. *if $\mathbb{V}$ is locally finite (and finitary) then $\widehat{T}X$ is the free profinite $\mathbb{V}$-algebra on X.*

Theorem

For a commutative and finitary monad T on Set*, the assignment $(X, M) \mapsto (\widehat{T}X, TM)$ yields a monad on BM.*

The semiring monads

Every semiring $(S, +, \cdot, 0, 1)$ induces a functor $\mathcal{S}\colon \mathrm{Set} \to \mathrm{Set}$ that sends a set X to

$$\mathcal{S}X := \{f\colon X \to S \mid f(x) = 0 \text{ for all but finitely many } x \in X\},$$

and a function $\psi\colon X \to Y$ to a function

$$\mathcal{S}\psi\colon \mathcal{S}X \to \mathcal{S}Y, \quad \sum_{i=1}^{n} s_i x_i \mapsto \sum_{i=1}^{n} s_i \psi(x_i).$$

In fact, $\mathcal{S}$ is a monad on Set (the semiring monad associated to S) whose algebras are modules over S.

Examples: $(2, \text{semilattices}), (\mathbb{N}, \text{Ab. monoids}), (\mathbb{Z}, \text{Ab. groups})$

Write $\widehat{\mathcal{S}} \colon \mathrm{BStone} \to \mathrm{BStone}$ for the profinite monad of S.

Theorem (Gehrke-Petrișan-R 2017)

Suppose Q is a quantifier modelled by a commutative semiring S, and let $\mathcal{S}$ be the associated monad on the category of sets. If the language $L_{\psi(x)}$ is recognised by a BM (X, M), then the quantified language $L_{Qx.\psi(x)}$ is recognised by the BM $(\Diamond X, \Diamond M) := (\widehat{\mathcal{S}}X \times X, \mathcal{S}M \times M)$.

Corollary

If the language $L_{\psi(x)}$ is recognised by (X, M), then the language $L_{\exists x.\psi(x)}$ is recognised by $(\mathcal{V}X \times X, \mathcal{P}_f M \times M)$.

Remark: the actions of the monoid $\Diamond M$ on the Boolean space $\Diamond X$ can be derived by duality. For $S = 2$ and X finite, they resemble the so-called Schützenberger product for monoids. Moreover,
$\phi \colon (\beta((A \times 2)^*), (A \times 2)^*) \to (X, M) \Rightarrow \Diamond\phi \colon (\beta(A^*), A^*) \to (\Diamond X, \Diamond M)$.

A Reutenauer-type result

The BM $(\Diamond X, \Diamond M)$ is optimal from the point of view of recognition:

Theorem (Gehrke-Petrișan-R 2017)

The Boolean subalgebra closed under quotients of $\mathcal{P}(A^*)$ generated by all languages recognised by some length-preserving morphism $(\beta(A^*), A^*) \to (\Diamond X, \Diamond M)$ is the BA generated by

$$\{L \subseteq A^* \mid L \text{ is recognised by } (X, M)\} \cup$$
$$\{\mathcal{Q}(L) \subseteq A^* \mid L \subseteq (A \times 2)^* \text{ is recognised by } (X, M)\}.$$

Measures

For finite and commutative S, we can explicitly describe $\widehat{\mathcal{S}}X$.

Lemma

Let $X \in \mathrm{BStone}$ and B its dual BA. The dual BA $\widehat{B}$ of $\widehat{\mathcal{S}}X$ is the subalgebra of $\mathcal{P}(\mathcal{S}X)$ generated by the elements of the form

$$[L, k] := \{f \in \mathcal{S}X \mid \sum_L f = k\}, \text{ for } L \in B, k \in S.$$

Every element of $\widehat{\mathcal{S}}X \cong BA(\widehat{B}, 2)$ induces a function $B \to S$:

$$(\widehat{B} \xrightarrow{\varphi} 2) \mapsto (\mu_\varphi \colon L \mapsto \text{unique } k \text{ s.t. } \varphi[L, k] = 1).$$

$\mu_\varphi \colon B \to S$ satisfies $\mu_\varphi(0) = 0$, and $\mu_\varphi(K \vee L) = \mu_\varphi(K) + \mu_\varphi(L)$ whenever $K \wedge L = 0$.

Measures

Definition

Let $X \in \mathrm{BStone}$ and B its dual BA. An S-valued measure on X is a function $\mu\colon B \to S$ s.t.

1. $\mu(0) = 0$;
2. $\mu(K \vee L) = \mu(K) + \mu(L)$ whenever $K \wedge L = 0$.

Equip the set of measures on X with the topology generated by

$$\{\mu\colon B \to S \mid \mu \text{ is a measure and } \mu(L) = k\}, \text{ for } L \in B, k \in S.$$

Theorem (Gehrke-Petrişan-R 2017)

For every $X \in \mathrm{BStone}$, $\varphi \mapsto \mu_\varphi$ is a homeomorphism between $\widehat{\mathcal{S}}X$ and the space of all S-valued measures on X.

Density functions

Suppose S is an idempotent, commutative and finite semiring (hence a semilattice with $x \leq y \Leftrightarrow x + y = y$ and $\vee = +$). Every measure $\mu \colon B \to S$ induces a (density) function

$$f_\mu \colon X \to S, \ x \mapsto \min \{\mu(L) \mid x \in L, \ L \in B\}$$

which is continuous w.r.t. the down-set topology on S.

Theorem

For every $X \in \mathrm{BStone}$, $\mu \mapsto f_\mu$ is a homeomorphism between $\widehat{\mathcal{S}}X$ and the space of all continuous functions from X to $S^{\downarrow}$.

Remark: for $S = 2$, this yields the usual representation of $\mathcal{V}X$ as the family of continuous functions from X into the Sierpiński space.

Thank you for your attention.