Quantifiers on languages and codensity monads

Luca Reggio Joint work with Mai Gehrke and Daniela Petrișan

IRIF, Université Paris Diderot, France

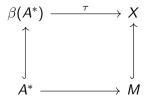
Topology, Algebra, and Categories in Logic 2017, Praha (June 26–30)

Topological recognisers: BMs

A Boolean space with an internal monoid (BM, or BiM, for short) is a pair (X, M) where

- X is a Boolean space;
- *M* is a dense subspace of *X* equipped with a monoid structure;
- the biaction of *M* on itself extends to a biaction of *M* on *X* with continuous components.

(injectivity assumption, in the general framework, has to be dropped)



First-order quantifiers

Some quantifiers we are interested in:

- existential quantifier ∃;
- modular quantifiers $\exists_{p \mod q}$. For $w \in (A \times 2)^*$, $w \models \exists_{p \mod q} x.\psi(x)$ iff there exist exactly $p \mod q$ positions in w for which the formula $\psi(x)$ is satisfied;
- semiring quantifiers $\exists_{k,S}$, for $(S, +, \cdot, 0_S, 1_S)$ a semiring and $k \in S$. If $w \in (A \times 2)^*$,

$$w \vDash \exists_{k,S} x. \psi(x) \Leftrightarrow \underbrace{1_{S} + \cdots + 1_{S}}_{m \text{ times}} = k,$$

where *m* is the number of positions in the word *w* that witness the validity of $\psi(x)$.

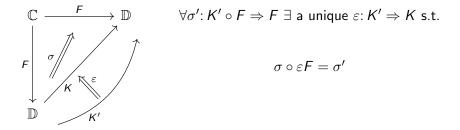
Question: Suppose (X, M) is a BM recognising the language $L_{\psi(x)}$. How to construct a BM recognising $L_{Qx,\psi(x)}$, for Q a certain (e.g. modular or semiring) quantifier?

[Gehrke-Petrişan-R 2016]: for $Q = \exists$, take $(\mathcal{V}X \times X, \mathcal{P}_f M \times M)$, where $\mathcal{V}X$ is the Vietoris space of X and $\mathcal{P}_f M$ is the finite powerset of M.

Hint for generalisation: $\mathcal{P}_f M$ is the free join-semilattice (=module over the two-element Boolean semiring) on M, and $\mathcal{V}X$ is the free profinite join-semilattice on X. In fact, \mathcal{V} is the profinite monad of \mathcal{P}_f .

Codensity and profinite monads

The codensity monad (Kock 60s) of a functor $F: \mathbb{C} \to \mathbb{D}$ is the monad on \mathbb{D} 'best approximating the monad that F would induce if it had a left adjoint'.



The pair (K, σ) is called the codensity monad of F. (Unit and multiplication of the monad by the universal property)

Codensity and profinite monads

If \mathbb{C} is (essentially) small and \mathbb{D} is complete, then $K: \mathbb{D} \to \mathbb{D}$ exists and is computed by $K(d) = \lim_{d \to F(c)} F(c)$.

Examples:

- 1. If $F: \operatorname{Set}_{fin} \hookrightarrow \operatorname{Set}$, then $K = \beta: \operatorname{Set} \to \operatorname{Set}$.
- 2. If $F: sLat_{fin} \rightarrow BStone$, then $K = \mathcal{V}: BStone \rightarrow BStone$.

If \mathbb{V} is the category of algebras for a monad T on Set, the profinite monad of T is the codensity monad of $\mathbb{V}_{fin} \to BStone$ (cf. item 2).

We will be interested in monads T that model a FO quantifier.

Let $T: \text{Set} \to \text{Set}$ be a monad and $\widehat{T}: \text{BStone} \to \text{BStone}$ its profinite monad. Write \mathbb{V} for the variety of T-algebras.

Lemma

For every Boolean space X, the following hold:

- 1. T|X| is dense in $\widehat{T}X$;
- 2. $\widehat{T}X$ is a profinite \mathbb{V} -algebra;
- 3. if \mathbb{V} is locally finite (and finitary) then $\widehat{T}X$ is the free profinite \mathbb{V} -algebra on X.

Theorem

For a commutative and finitary monad T on Set, the assignment $(X, M) \mapsto (\widehat{T}X, TM)$ yields a monad on BM.

The semiring monads

Every semiring $(S, +, \cdot, 0, 1)$ induces a functor \mathcal{S} : Set o Set that sends a set X to

 $\mathcal{S}X := \{f \colon X \to S \mid f(x) = 0 \text{ for all but finitely many } x \in X\},\$

and a function $\psi: X \to Y$ to a function

$$S\psi: SX \to SY, \quad \sum_{i=1}^n s_i x_i \mapsto \sum_{i=1}^n s_i \psi(x_i).$$

In fact, S is a monad on Set (the semiring monad associated to S) whose algebras are modules over S.

Examples: (2, semilattices), (\mathbb{N} , Ab. monoids), (\mathbb{Z} , Ab. groups)

Luca Reggio

Write \widehat{S} : BStone \rightarrow BStone for the profinite monad of *S*.

Theorem (Gehrke-Petrişan-R 2017)

Suppose Q is a quantifier modelled by a commutative semiring S, and let S be the associated monad on the category of sets. If the language $L_{\psi(x)}$ is recognised by a BM (X, M), then the quantified language $L_{Qx,\psi(x)}$ is recognised by the BM $(\Diamond X, \Diamond M) := (\widehat{S}X \times X, SM \times M)$.

Corollary

If the language $L_{\psi(x)}$ is recognised by (X, M), then the language $L_{\exists x.\psi(x)}$ is recognised by $(\mathcal{V}X \times X, \mathcal{P}_f M \times M)$.

Remark: the actions of the monoid $\Diamond M$ on the Boolean space $\Diamond X$ can be derived by duality. For S = 2 and X finite, they resemble the so-called Schützenberger product for monoids. Moreover, $\phi: (\beta((A \times 2)^*), (A \times 2)^*) \to (X, M) \Rightarrow \Diamond \phi: (\beta(A^*), A^*) \to (\Diamond X, \Diamond M).$

A Reutenauer-type result

The BM ($\Diamond X, \Diamond M$) is optimal from the point of view of recognition:

Theorem (Gehrke-Petrişan-R 2017)

The Boolean subalgebra closed under quotients of $\mathcal{P}(A^*)$ generated by all languages recognised by some length-preserving morphism $(\beta(A^*), A^*) \rightarrow (\Diamond X, \Diamond M)$ is the BA generated by

 $\{L \subseteq A^* \mid L \text{ is recognised by } (X, M)\} \cup \\ \{Q(L) \subseteq A^* \mid L \subseteq (A \times 2)^* \text{ is recognised by } (X, M)\}.$

Measures

For finite and commutative S, we can explicitly describe \widehat{SX} .

Lemma

Let $X \in B$ Stone and B its dual BA. The dual $BA \ \widehat{B}$ of $\widehat{S}X$ is the subalgebra of $\mathcal{P}(SX)$ generated by the elements of the form

$$[L,k] := \{f \in \mathcal{S}X \mid \sum_{L} f = k\}, \text{ for } L \in B, k \in \mathcal{S}.$$

Every element of $\widehat{S}X \cong BA(\widehat{B},2)$ induces a function $B \to S$:

$$(\widehat{B} \xrightarrow{\varphi} 2) \mapsto (\mu_{\varphi}: L \mapsto \text{unique } k \text{ s.t. } \varphi[L, k] = 1).$$

 $\mu_{\varphi}: B \to S$ satisfies $\mu_{\varphi}(0) = 0$, and $\mu_{\varphi}(K \vee L) = \mu_{\varphi}(K) + \mu_{\varphi}(L)$ whenever $K \wedge L = 0$.

Measures

Definition

Let $X \in BS$ tone and B its dual BA. An *S*-valued measure on X is a function $\mu: B \to S$ s.t.

1. $\mu(0) = 0;$ 2. $\mu(K \lor L) = \mu(K) + \mu(L)$ whenever $K \land L = 0.$

Equip the set of measures on X with the topology generated by

$$\{\mu: B \to S \mid \mu \text{ is a measure and } \mu(L) = k\}, \text{ for } L \in B, k \in S.$$

Theorem (Gehrke-Petrişan-R 2017)

For every $X \in BStone$, $\varphi \mapsto \mu_{\varphi}$ is a homeomorphism between $\widehat{S}X$ and the space of all S-valued measures on X.

Density functions

Suppose S is an idempotent, commutative and finite semiring (hence a semilattice with $x \le y \Leftrightarrow x + y = y$ and $\lor = +$). Every measure $\mu: B \to S$ induces a (density) function

$$f_{\mu}: X \to S, x \mapsto \min \{\mu(L) \mid x \in L, L \in B\}$$

which is continuous w.r.t. the down-set topology on S.

Theorem

For every $X \in BStone$, $\mu \mapsto f_{\mu}$ is a homeomorphism between $\widehat{S}X$ and the space of all continuous functions from X to S^{\downarrow} .

Remark: for S = 2, this yields the usual representation of VX as the family of continuous functions from X into the Sierpiński space.

Thank you for your attention.