

Modular proof theory for axiomatic extension and expansions of lattice logic

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Multi-type methodology

Syntax meets semantics: the wider picture

Multi-type (algebraic) proof theory

- ▶ constructive canonical extensions algebra, formal topology
- ▶ unified correspondence theory duality
- ▶ proper display calculi structural proof theory

Proof calculi with a uniform metatheory:

- ▶ supporting an **inferential theory of meaning**
- ▶ canonical **cut elimination** and **subformula property**
- ▶ **soundness, completeness, conservativity**

Range

- ▶ DEL, PDL, Logic of resources and capabilities...
- ▶ (D)LEs and their analytic inductive axiomatic extensions
- ▶ Inquisitive logic, first order logic
- ▶ Linear logic
- ▶ Lattice logic ! / Modular lattice logic !?

Intermezzo on proof theory

Hilbert Calculi

- ▶ Axioms (E. Mendelson):

- (A1) $p \rightarrow (q \rightarrow p)$

- (A2) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

- (A3) $(\neg p \rightarrow \neg q) \rightarrow ((\neg p \rightarrow q) \rightarrow p)$

- ▶ Rules: US, MP

Hilbert Calculi

- Axioms (E. Mendelson):

(A1) $p \rightarrow (q \rightarrow p)$

(A2) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

(A3) $(\neg p \rightarrow \neg q) \rightarrow ((\neg p \rightarrow q) \rightarrow p)$

- ▶ Rules: US, MP

$$1 \quad (A \rightarrow ((A \rightarrow A) \rightarrow A)) \rightarrow ((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))$$

$$2 \quad A \rightarrow ((A \rightarrow A) \rightarrow A)$$

$$3 \quad (A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A)$$

$$4 \quad A \rightarrow (A \rightarrow A)$$

5 $A \rightarrow A$

$$\frac{\frac{(A2)}{1} \text{ US}[A/p, A \rightarrow A/q, A/r] \quad \frac{(A1)}{2} \text{ US}[A/p, A \rightarrow A/q, A/r]}{3} \text{ MP} \quad \frac{(A1)}{4} \text{ US}[A/p, A/q]}{5} \text{ MP}$$

Starting point: Display Calculi

- ▶ Natural generalization of Gentzen's sequent calculi;
- ▶ sequents $X \vdash Y$, where X and Y are **structures**:
 - formulas are **atomic structures**
 - built-up: **structural connectives** (generalizing meta-linguistic comma in sequents $\phi_1, \dots, \phi_n \vdash \psi_1, \dots, \psi_m$)
 - generation **trees** (generalizing sets, multisets, sequences)
- ▶ **Display property**:

$$\frac{\frac{\frac{Y \vdash X > Z}{X ; Y \vdash Z}}{Y ; X \vdash Z}}{X \vdash Y > Z}$$

display rules semantically justified by **adjunction/residuation**

- ▶ **Canonical proof of cut elimination (via metatheorem)**

Structural and operational languages

$$A ::= p \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid (A \multimap A) \mid \neg A$$
$$X ::= A \mid I \mid X ; X \mid X > X$$

Structural connectives are interpreted ***positionally***

(like Gentzen's comma) :

I		;		>		*	
$\top$	$\perp$	$\wedge$	$\vee$	$(\multimap)$	$\rightarrow$	$\neg$	$\neg$

Three groups of rules

Display Postulates

$$\frac{X; Y \vdash Z}{Y \vdash X > Z} \quad \frac{Z \vdash Y; X}{Y > Z \vdash X}$$

Operational Rules

$$\frac{A; B \vdash X}{A \wedge B \vdash X} \quad \frac{X \vdash A \quad Y \vdash B}{X; Y \vdash A \wedge B}$$
$$\frac{X \vdash A \quad B \vdash Y}{A \rightarrow B \vdash X > Y} \quad \frac{X \vdash A > B}{X \vdash A \rightarrow B}$$

Structural Rules

$$Gri_L \frac{(X > Y); Z \vdash W}{X > (Y; Z) \vdash W} \quad \frac{W \vdash (X > Y); Z}{W \vdash X > (Y; Z)} Gri_R$$

The excluded middle is derivable using *Grishin's rule*:

$$\begin{array}{c}
 \frac{A \vdash A}{A ; I \vdash A} \\
 \frac{A ; I \vdash A}{A ; I \vdash \perp ; A} \\
 \frac{A ; I \vdash \perp ; A}{I \vdash A > (\perp ; A)} \\
 \frac{I \vdash A > (\perp ; A)}{I \vdash (A > \perp) ; A} \text{ Gri} \\
 \frac{I \vdash (A > \perp) ; A}{I \vdash A ; (A > \perp)} \\
 \frac{A > I \vdash A > \perp}{A > I \vdash A \rightarrow \perp} \\
 \frac{A > I \vdash A \rightarrow \perp}{A > I \vdash \neg A} \\
 \frac{A > I \vdash \neg A}{I \vdash A ; \neg A} \\
 \frac{I \vdash A ; \neg A}{I \vdash A \vee \neg A}
 \end{array}$$

Cut rules in Gentzen's Calculi

$$\begin{array}{c} \frac{\Gamma \vdash C, \Delta \quad \Gamma', C \vdash \Delta'}{\Gamma', \Gamma \vdash \Delta', \Delta} \quad \frac{\Gamma \vdash C, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta} \quad \frac{\Gamma \vdash C \quad \Gamma, C \vdash \Delta}{\Gamma \vdash \Delta} \\[10pt] \frac{\Gamma \vdash C \quad \Gamma', C \vdash \Delta}{\Gamma', \Gamma \vdash \Delta} \quad \frac{\Gamma \vdash C, \Delta \quad C \vdash \Delta'}{\Gamma \vdash \Delta', \Delta} \quad \frac{\Gamma \vdash C \quad C \vdash \Delta}{\Gamma \vdash \Delta} \end{array}$$

Theorem

If $\Gamma \vdash \Delta$ is derivable, then it is derivable without Cut.

- ✓ A Cut is an intermediate step in a deduction.
'Eliminating the cut' generates a **new and lemma-free proof**, which employs **syntactic material coming from the end-sequent**.
- ✗ Typically, syntactic proofs of Cut-elimination are **non-modular**,
i.e. if a new rule is added, it must be proved from scratch.

Multi-type proper display calculi

Definition

A **proper display calculus** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation (same shape, position, non-proliferation)
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters **within each type (Properness!)**;
5. **reduction strategy** exists when cut formulas are principal.
6. **type-uniformity** of derivable sequents;
7. **strongly uniform cuts** in each/some type(s).

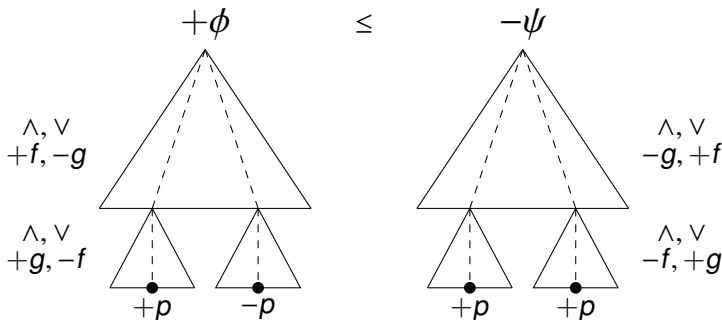
Theorem (Canonical!)

Cut elimination and subformula property hold for any **proper display calculus**.

Which logics are properly displayable?

Complete characterization (Ciabattoni et al. 15, Greco et al. 16):

1. the logics of any **basic** normal (D)LE;
2. axiomatic extensions of these with **analytic inductive inequalities**: $\rightsquigarrow$ **unified correspondence**



Analytic inductive $\Rightarrow$ Inductive $\Rightarrow$ Canonical

Fact: cut-elim., subfm. prop., sound-&-completeness, conservativity **guaranteed** by metatheoem + ALBA-technology.

For many... but not for all.

- ▶ The characterization theorem sets **hard boundaries** to the scope of proper display calculi.
- ▶ Interesting logics are **left out**.

Can we **extend the scope** of proper display calculi?

Yes: proper display calculi $\rightsquigarrow$ proper **multi-type** calculi

Lattice logic

Is Lattice Logic properly displayable?

$$\frac{A \vdash X}{A \wedge B \vdash X} \quad \frac{B \vdash X}{A \wedge B \vdash X} \quad \frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B}$$
$$\frac{A \vdash X \quad B \vdash X}{A \vee B \vdash X} \quad \frac{X \vdash A}{X \vdash A \vee B} \quad \frac{X \vdash A}{X \vdash A \vee B}$$

In general lattices, $\wedge$ and $\vee$ are **adjoints** but not residuals.

[Belnap 92, Sambin et al 00]: no structural counterparts.

Remark: rules $\wedge_R$ and $\vee_R$ encode $\vee \dashv \Delta \dashv \wedge$.

What is wrong with this solution?

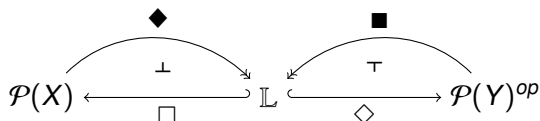
Nothing: as a "Gentzen" calculus, it is perfectly fine.

However: an **imbalance**

- ▶ **too much** information encoded in logical rules
 - ▶ introduction rules as **adjunction rules**
- ▶ **too little** information encoded in structural rules
 - ▶ no structural counterparts of $\wedge$ and $\vee$, hence
 - no structural rules capturing the behaviour of $\wedge$ and $\vee$
 - no interaction between $\wedge$ and $\vee$ and other connectives

Exception to a completely modular and uniform theory.

Algebraic analysis: double representation



Representation theorem

Any complete lattice $\mathbb{L}$ can be identified with:

- ▶ complete sub $\cap$ -semilattice of some $\mathcal{P}(X)$;
- ▶ complete sub $\cup$ -semilattice of some $\mathcal{P}(Y)^{op}$.

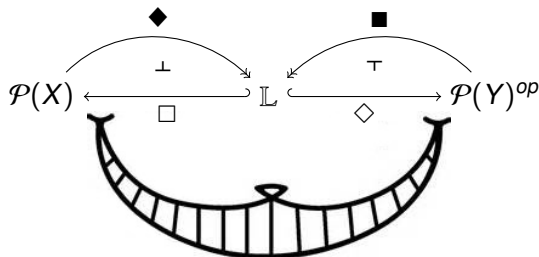
Upshot: natural semantics for the following **multi-type** language:

Left $\ni \alpha ::= \square A \mid \wp \mid \emptyset \mid \alpha \cap \alpha \mid \alpha \cup \alpha$

Lattice $\ni A ::= p \mid \top \mid \perp \mid \blacklozenge \alpha \mid \blacksquare \xi$

Right $\ni \xi ::= \lozenge A \mid \wp^{op} \mid \emptyset^{op} \mid \xi \cap^{op} \xi \mid \xi \cup^{op} \xi$

Translation



[...] this time it vanished quite slowly,

beginning with the end of the tail,

and ending with the grin, which remained

some time after the rest of it had gone.

$$A \vdash B \rightsquigarrow A^\tau \vdash B_\tau$$

$$\top^\tau := \blacklozenge \square \top$$

$$\perp^\tau := \blacklozenge \square \perp$$

$$p^\tau := \blacklozenge \square p$$

$$(A \wedge B)^\tau := \blacklozenge (\square A^\tau \cap \square B^\tau)$$

$$(A \vee B)^\tau := \blacklozenge (\square A^\tau \cup \square B^\tau)$$

$$\top_\tau := \blacksquare^{op} \diamond^{op} \top$$

$$\perp_\tau := \blacksquare^{op} \diamond^{op} \perp$$

$$p_\tau := \blacksquare^{op} \diamond^{op} p$$

$$(A \wedge B)_\tau := \blacksquare^{op} (\diamond^{op} A_\tau \cap \diamond^{op} B_\tau)$$

$$(A \vee B)_\tau := \blacksquare^{op} (\diamond^{op} A_\tau \cup \diamond^{op} B_\tau)$$

Proper multi-type calculus for lattice logic - Part 1

Display Postulates

$$\text{adj} \frac{\Gamma \vdash \circ X}{\bullet \Gamma \vdash X}$$

$$\frac{\circ X \vdash \Pi}{X \vdash \bullet \Pi} \text{adj}$$

$$\text{res} \frac{\Gamma . \Delta \vdash \Lambda}{\Delta \vdash \Gamma \supset \Lambda}$$

$$\frac{\Gamma \vdash \Delta . \Lambda}{\Delta \supset \Gamma \vdash \Lambda} \text{res}$$

$$\text{res} \frac{\Pi . \Upsilon \vdash \Omega}{\Upsilon \vdash \Pi \supset \Omega}$$

$$\frac{\Pi \vdash \Upsilon . \Omega}{\Upsilon \supset \Pi \vdash \Omega} \text{res}$$

Proper multi-type calculus for lattice logic - Part 2

Lattice rules

$$Id \frac{}{p \vdash p} \quad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y} Cut$$

$$IW \frac{I \vdash X}{Y \vdash X} \quad \frac{X \vdash I}{X \vdash Y} IW$$

$$\top \frac{I \vdash X}{\top \vdash X} \quad \frac{}{I \vdash \top} \top$$

$$\perp \frac{}{\perp \vdash I} \quad \frac{X \vdash I}{X \vdash \perp} \perp$$

Identity Lemma

Lemma: The sequent $A^\tau \vdash A_\tau$ is derivable for every $A \in \mathcal{L}$.

$$\begin{array}{c}
\text{Id} \\
\frac{p \vdash p}{\square p \vdash \circ p} \\
\text{adj} \frac{\bullet \square p \vdash p}{\blacklozenge \square p \vdash p} \\
\frac{\blacklozenge \square p \vdash p}{\circ \blacklozenge \square p \vdash \blacklozenge p} \\
\text{adj} \frac{\blacklozenge \square p \vdash \bullet \blacklozenge p}{\blacklozenge \square p \vdash \blacksquare \blacklozenge p}
\end{array}
\qquad
\begin{array}{c}
\text{ind. hyp.} \\
\frac{B^\tau \vdash B_\tau}{\square B^\tau \vdash \circ B_\tau} \\
W \frac{\square B^\tau \vdash \circ B_\tau}{\square B^\tau \cdot \square C^\tau \vdash \circ B_\tau} \\
\frac{\square B^\tau \cdot \square C^\tau \vdash \circ B_\tau}{\square B^\tau \cap \square C^\tau \vdash \circ B_\tau} \\
\frac{\square B^\tau \cap \square C^\tau \vdash \circ B_\tau}{\bullet \square B^\tau \cap \square C^\tau \vdash B_\tau} \\
\frac{\bullet \square B^\tau \cap \square C^\tau \vdash B_\tau}{\blacklozenge (\square B^\tau \cap \square C^\tau) \vdash B_\tau} \\
\frac{\blacklozenge (\square B^\tau \cap \square C^\tau) \vdash B_\tau}{\circ \blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \blacklozenge B_\tau} \\
C \frac{\circ \blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \blacklozenge B_\tau}{\circ \blacklozenge (\square B^\tau \cap \square C^\tau) \cdot \circ \blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \blacklozenge B_\tau \cap \blacklozenge C_\tau} \\
\frac{\circ \blacklozenge (\square B^\tau \cap \square C^\tau) \cdot \circ \blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \blacklozenge B_\tau \cap \blacklozenge C_\tau}{\blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \bullet \blacklozenge B_\tau \cap \blacklozenge C_\tau} \\
\text{adj} \frac{\blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \bullet \blacklozenge B_\tau \cap \blacklozenge C_\tau}{\blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \blacksquare (\blacklozenge B_\tau \cap \blacklozenge C_\tau)}
\end{array}
\qquad
\begin{array}{c}
\text{ind. hyp.} \\
\frac{C^\tau \vdash C_\tau}{\square C^\tau \vdash \circ C_\tau} \\
W \frac{\square C^\tau \vdash \circ C_\tau}{\square C^\tau \cdot \square B^\tau \vdash \circ C_\tau} \\
E \frac{\square C^\tau \cdot \square B^\tau \vdash \circ C_\tau}{\square B^\tau \cdot \square C^\tau \vdash \circ C_\tau} \\
\frac{\square B^\tau \cdot \square C^\tau \vdash \circ C_\tau}{\square B^\tau \cap \square C^\tau \vdash \circ C_\tau} \\
\frac{\square B^\tau \cap \square C^\tau \vdash \circ C_\tau}{\bullet \square B^\tau \cap \square C^\tau \vdash C_\tau} \\
\frac{\bullet \square B^\tau \cap \square C^\tau \vdash C_\tau}{\blacklozenge (\square B^\tau \cap \square C^\tau) \vdash C_\tau} \\
\frac{\blacklozenge (\square B^\tau \cap \square C^\tau) \vdash C_\tau}{\circ \blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \blacklozenge C_\tau} \\
\frac{\circ \blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \blacklozenge C_\tau}{\circ \blacklozenge (\square B^\tau \cap \square C^\tau) \cdot \circ \blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \blacklozenge B_\tau \cap \blacklozenge C_\tau} \\
\text{adj} \frac{\circ \blacklozenge (\square B^\tau \cap \square C^\tau) \cdot \circ \blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \blacklozenge B_\tau \cap \blacklozenge C_\tau}{\blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \bullet \blacklozenge B_\tau \cap \blacklozenge C_\tau} \\
\frac{\blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \bullet \blacklozenge B_\tau \cap \blacklozenge C_\tau}{\blacklozenge (\square B^\tau \cap \square C^\tau) \vdash \blacksquare (\blacklozenge B_\tau \cap \blacklozenge C_\tau)}
\end{array}$$

Commutativity derived

$$\begin{array}{c}
 B \vdash B \\
 \hline
 \vdash \\
 \hline
 W \vdash B \cdot A \vdash B \\
 E \vdash A \cdot B \vdash B \\
 \hline
 A \cap B \vdash B \\
 \hline
 \vdash \\
 \hline
 \vdash \\
 \hline
 \vdash \\
 \hline
 C \vdash (A \cap B) \cdot (A \cap B) \vdash B \cap A \\
 \hline
 A \cap B \vdash B \cap A \\
 \hline
 \vdash \\
 \hline
 \vdash
 \end{array}$$

$$\begin{array}{c}
 A \vdash A \\
 \hline
 \vdash \\
 \hline
 W \vdash A \cdot B \vdash A \\
 \hline
 A \cap B \vdash A \\
 \hline
 \vdash \\
 \hline
 \vdash \\
 \hline
 \vdash \\
 \hline
 \vdash
 \end{array}$$

Translation of commutativity derived

$$\begin{array}{c}
 \frac{B^\tau \vdash B_\tau}{\square B^\tau \vdash \circ B_\tau} \\
 \frac{W}{\square B^\tau \cdot \square A^\tau \vdash \circ B_\tau} \\
 \frac{E}{\square A^\tau \cdot \square B^\tau \vdash \circ B_\tau} \\
 \frac{}{\square A^\tau \cap \square B^\tau \vdash \circ B_\tau} \\
 \frac{}{\bullet \square A^\tau \cap \square B^\tau \vdash B_\tau} \\
 \frac{}{\blacklozenge(\square A^\tau \cap \square B^\tau) \vdash B_\tau} \\
 \frac{}{\circ \blacklozenge(\square A^\tau \cap \square B^\tau) \vdash \blacklozenge B_\tau} \\
 \frac{}{\circ \blacklozenge(\square A^\tau \cap \square B^\tau) \cdot \circ \blacklozenge(\square A^\tau \cap \square B^\tau) \vdash \blacklozenge B_\tau \cap \blacklozenge A_\tau} \\
 C \frac{}{}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{A^\tau \vdash A_\tau}{\square A^\tau \vdash \circ A_\tau} \\
 \frac{W}{\square A^\tau \cdot \square B^\tau \vdash \circ A_\tau} \\
 \frac{}{\square A^\tau \cap \square B^\tau \vdash \circ A_\tau} \\
 \frac{}{\bullet \square A^\tau \cap \square B^\tau \vdash A_\tau} \\
 \frac{}{\blacklozenge(\square A^\tau \cap \square B^\tau) \vdash A_\tau} \\
 \frac{}{\circ \blacklozenge(\square A^\tau \cap \square B^\tau) \vdash \blacklozenge A_\tau} \\
 \frac{}{\circ \blacklozenge(\square A^\tau \cap \square B^\tau) \vdash \blacklozenge B_\tau \cap \blacklozenge A_\tau} \\
 \frac{}{\blacklozenge(\square A^\tau \cap \square B^\tau) \vdash \bullet \blacklozenge B_\tau \cap \blacklozenge A_\tau} \\
 \frac{}{\blacklozenge(\square A^\tau \cap \square B^\tau) \vdash \blacksquare(\blacklozenge B_\tau \cap \blacklozenge A_\tau)}
 \end{array}$$

Translation of Identity derived ($A := p$)

$$\begin{array}{c}
 \text{Id Lemma} \\
 \hline
 \blacklozenge \Box p \vdash \blacksquare \Diamond p \\
 \hline
 \Box \blacklozenge \Box p \vdash \circ \blacksquare \Diamond p \\
 \hline
 w \frac{\Box \blacklozenge \Box p \cdot \Box \blacklozenge \Box T \vdash \circ \blacksquare \Diamond p}{\Box \blacklozenge \Box p \cap \Box \blacklozenge \Box T \vdash \circ \blacksquare \Diamond p} \\
 \hline
 \bullet \Box \blacklozenge \Box p \cap \Box \blacklozenge \Box T \vdash \blacksquare \Diamond p \\
 \hline
 \blacklozenge (\Box \blacklozenge \Box p \cap \Box \blacklozenge \Box T) \vdash \blacksquare \Diamond p
 \end{array}$$

$$\begin{array}{c}
 \text{Id Lemma} \\
 \hline
 \blacklozenge \Box p \vdash \blacksquare \Diamond p \\
 \hline
 \circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p \\
 \hline
 +C \frac{\circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p}{\circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p} \\
 \hline
 C \frac{\circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p \cap \Diamond \blacksquare \Diamond T}{\circ \blacklozenge \Box p \vdash \bullet \Diamond \blacksquare \Diamond p \cap \Diamond \blacksquare \Diamond T} \\
 \hline
 \blacklozenge \Box p \vdash \bullet \Diamond \blacksquare \Diamond p \cap \Diamond \blacksquare \Diamond T \\
 \hline
 \blacklozenge \Box p \vdash \blacksquare (\Diamond \blacksquare \Diamond p \cap \Diamond \blacksquare \Diamond T)
 \end{array}$$

$$\begin{array}{c}
 \text{IW} \frac{I \vdash T}{\blacklozenge \Box p \vdash T} \\
 \hline
 \circ \blacklozenge \Box p \vdash \Diamond T \\
 \hline
 \blacklozenge \Box p \vdash \bullet \Diamond T \\
 \hline
 \blacklozenge \Box p \vdash \blacksquare \Diamond T \\
 \hline
 \circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond T \\
 \hline
 \text{Id Lemma} \\
 \hline
 \blacklozenge \Box p \vdash \blacksquare \Diamond p \\
 \hline
 \circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p \\
 \hline
 \circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p \cap \Diamond \blacksquare \Diamond T \\
 \hline
 \circ \blacklozenge \Box p \vdash \bullet \Diamond \blacksquare \Diamond p \cap \Diamond \blacksquare \Diamond T \\
 \hline
 \blacklozenge \Box p \vdash \bullet \Diamond \blacksquare \Diamond p \cap \Diamond \blacksquare \Diamond T \\
 \hline
 \blacklozenge \Box p \vdash \blacksquare (\Diamond \blacksquare \Diamond p \cap \Diamond \blacksquare \Diamond T)
 \end{array}$$

Translation of Absorption derived (case $A := p$)

$$\begin{array}{c}
 \text{Id Lemma} \\
 \frac{\blacklozenge \Box p \vdash \blacksquare \Diamond p}{\Box \blacklozenge \Box p \vdash \circ \blacksquare \Diamond p} \\
 W \frac{\Box \blacklozenge \Box p \vdash \circ \blacksquare \Diamond p}{\Box \blacklozenge \Box p \vdash \circ \blacksquare \Diamond p} \\
 \frac{\Box \blacklozenge \Box p \vdash \circ \blacksquare \Diamond p}{\Box \blacklozenge \Box p \vdash \circ \blacksquare \Diamond p} \\
 \frac{\bullet \Box \blacklozenge \Box p \vdash \blacksquare \Diamond p}{\blacklozenge (\Box \blacklozenge \Box p \vdash \blacksquare \Diamond p)} \\
 \frac{\blacklozenge (\Box \blacklozenge \Box p \vdash \blacksquare \Diamond p)}{\blacklozenge (\Box \blacklozenge \Box p \vdash \blacksquare \Diamond p)}
 \end{array}$$

$$\begin{array}{c}
 \text{Id Lemma} \\
 \frac{\blacklozenge \Box p \vdash \blacksquare \Diamond p}{\circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p} \\
 W \frac{\circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p}{\circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p \cup \Diamond B} \\
 \frac{\circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p \cup \Diamond B}{\blacklozenge \Box p \vdash \bullet \Diamond \blacksquare \Diamond p \cup \Diamond B} \\
 \frac{\blacklozenge \Box p \vdash \bullet \Diamond \blacksquare \Diamond p \cup \Diamond B}{\blacklozenge \Box p \vdash \blacksquare (\Diamond \blacklozenge \Diamond p \cup \Diamond B)} \\
 \frac{\blacklozenge \Box p \vdash \blacksquare (\Diamond \blacklozenge \Diamond p \cup \Diamond B)}{\circ \blacklozenge \Box p \vdash \Diamond \blacksquare (\Diamond \blacklozenge \Diamond p \cup \Diamond B)}
 \end{array}$$

$$\begin{array}{c}
 \text{Id Lemma} \\
 \frac{\blacklozenge \Box p \vdash \blacksquare \Diamond p}{\circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p} \\
 C \frac{\circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p}{\circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p \cap \Diamond \blacksquare (\Diamond \blacklozenge \Diamond p \cup \Diamond B)} \\
 \frac{\circ \blacklozenge \Box p \vdash \Diamond \blacksquare \Diamond p \cap \Diamond \blacksquare (\Diamond \blacklozenge \Diamond p \cup \Diamond B)}{\blacklozenge \Box p \vdash \bullet \Diamond \blacksquare \Diamond p \cap \Diamond \blacksquare (\Diamond \blacklozenge \Diamond p \cup \Diamond B)} \\
 \frac{\blacklozenge \Box p \vdash \bullet \Diamond \blacksquare \Diamond p \cap \Diamond \blacksquare (\Diamond \blacklozenge \Diamond p \cup \Diamond B)}{\blacklozenge \Box p \vdash \blacksquare (\Diamond \blacklozenge \Diamond p \cap \Diamond \blacksquare (\Diamond \blacklozenge \Diamond p \cup \Diamond B))}
 \end{array}$$

Proof for a distributive lattice

$$\begin{array}{c}
 \frac{s \vdash s \quad t \vdash t}{s.t \vdash s \cap t} \quad \text{res} \qquad \frac{s \vdash s \quad u \vdash u}{s.u \vdash s \cap u} \quad \text{res} \\
 \frac{t \vdash s \supset s \cap t \quad s.u \vdash s \cap u}{t \cup u \vdash (s \supset s \cap t) \cdot (s \supset s \cap u)} \quad \text{int-Gri} \\
 \frac{t \cup u \vdash s \supset (s \cap t \cdot (s \supset s \cap u))}{s.t \cup u \vdash s \cap t \cdot (s \supset s \cap u)} \quad \text{res} \\
 \frac{s.t \cup u \vdash (s \supset s \cap u) \cdot s \cap t}{s.t \cup u \vdash s \supset (s \cap u \cdot s \cap t)} \quad \text{int-Gri} \\
 \frac{s.t \cup u \vdash s \supset (s \cap u \cdot s \cap t)}{s.(s.t \cup u) \vdash s \cap u \cdot s \cap t} \quad \text{res} \\
 \frac{s.(s.t \cup u) \vdash (s \cap t) \cup (s \cap u)}{(s.s).t \cup u \vdash (s \cap t) \cup (s \cap u)} \\
 \frac{(s.s).t \cup u \vdash (s \cap t) \cup (s \cap u)}{t \cup u.(s.s) \vdash (s \cap t) \cup (s \cap u)} \\
 \frac{t \cup u.(s.s) \vdash (s \cap t) \cup (s \cap u)}{C \quad \frac{s.s \vdash t \cup u \supset (s \cap t) \cup (s \cap u)}{s \vdash t \cup u \supset (s \cap t) \cup (s \cap u)}} \\
 \frac{s \vdash t \cup u \supset (s \cap t) \cup (s \cap u)}{t \cup u.s \vdash (s \cap t) \cup (s \cap u)} \\
 \frac{t \cup u.s \vdash (s \cap t) \cup (s \cap u)}{s.t \cup u \vdash (s \cap t) \cup (s \cap u)} \\
 \frac{s.t \cup u \vdash (s \cap t) \cup (s \cap u)}{s \cap (t \cup u) \vdash (s \cap t) \cup (s \cap u)}
 \end{array}$$

Distributivity fails

$$\begin{array}{c}
 \frac{\frac{s \vdash s}{\circ s \vdash \Diamond s} \quad \frac{t \vdash t}{\circ t \vdash \Diamond t}}{\circ s . \circ t \vdash \Diamond s \cap \Diamond t} \\
 \frac{\circ t \vdash \circ s \supset \Diamond s \cap \Diamond t}{\circ t \vdash \circ s \supset \Diamond s \cap \Diamond t} \\
 \frac{t \vdash \bullet(\circ s \supset \Diamond s \cap \Diamond t)}{\Box t \vdash \circ \bullet(\circ s \supset \Diamond s \cap \Diamond t)} \\
 \hline
 \Box t \cup \Box u \vdash \circ \bullet(\circ s \supset \Diamond s \cap \Diamond t) . \circ \bullet(\circ s \supset \Diamond s \cap \Diamond u) \\
 \\
 \vdots \\
 ???
 \end{array}$$

Modularity

Every distributive lattice is modular.

Modular lattice: $c \leq b$ implies $c \vee (a \wedge b) = (c \vee a) \wedge b$.

In every lattice: $c \leq b$ implies $c \vee (a \wedge b) \leq (c \vee a) \wedge b$.

$$\begin{array}{c}
 \frac{c \vdash c \quad b \vdash b}{c, b \vdash c \wedge b} \\
 \frac{c, b \vdash c \wedge b, a}{c, b \vdash (c \wedge b) \vee a} \quad w \\
 \frac{c, b \vdash (c \wedge b) \vee a \quad b \vdash b}{c, b \vdash ((c \wedge b) \vee a) \wedge b} \\
 \frac{c \wedge b \vdash ((c \wedge b) \vee a) \wedge b}{(c \wedge b) \vee (a \wedge b) \vdash ((c \wedge b) \vee a) \wedge b, ((c \wedge b) \vee a) \wedge b} \\
 \frac{(c \wedge b) \vee (a \wedge b) \vdash ((c \wedge b) \vee a) \wedge b, ((c \wedge b) \vee a) \wedge b}{(c \wedge b) \vee (a \wedge b) \vdash ((c \wedge b) \vee a) \wedge b} \quad c
 \end{array}$$

In between “visibility” and “display” property

$$\frac{\frac{\frac{c \vdash c \quad b \vdash b}{c, b \vdash c \wedge b}}{c \wedge b \vdash c \wedge b} \quad \frac{a \vdash a \quad b \vdash b}{a, b \vdash a \wedge b}}{\frac{\vee \quad (c \wedge b) \vee a, b \vdash (c \wedge b), (a \wedge b)}{(c \wedge b) \vee a, b \vdash (c \wedge b) \vee (a \wedge b)}} \\ \frac{}{((c \wedge b) \vee a) \wedge b \vdash (c \wedge b) \vee (a \wedge b)}$$

Beyond analiticity: towards a general theory

- ▶ Several examples of logics which are **single-type not analytic** but **multi-type analytic**:
 - ▶ DEL
 - ▶ inquisitive logic
 - ▶ (intuitionistic modal) dependence logic
 - ▶ linear logic
 - ▶ lattice logic
 - ▶ PDL
 - ▶ logic of resources and capabilities
 - ▶ first order logic
 - ▶ ...
- ▶ Patterns are emerging. Main guideline: discovering and exploiting hidden adjunctions.
- ▶ Can we make this practice into a uniform theory?

- ▶ Frittella, Greco, Kurz, Palmigiano, [Multi-Type Sequent Calculi](#), Proc. Trends in Logics, 2014.
- ▶ Greco, Ma, Palmigiano, Tzimoulis, Zhao, [Unified Correspondence as a Proof-Theoretic Tool](#), JLC, 2016.
- ▶ Ciabattoni, Ramanayake, [Power and limits of structural display rules](#), TOCL, 2016.
- ▶ Greco, Palmigiano, [Linear Logic Properly Displayed](#), ArXiv: 1611.04181.