Bimodal bilattice logic

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Outline

Lattice-valued modal logics

Defined in terms of $\mathbf{A}$**-valued Kripke models** for a lattice $\mathbf{A}$, $M = \langle W, R, e \rangle$

- W is a non-empty set
- R is a function from $W \times W$ to $\mathbf{A}$
- e is a function from $Fm_0 \times W$ to $\mathbf{A}$

The value of $\Box\varphi$ at w is defined in terms of the lattice-order **infimum** of values related to φ.

M is called **crisp** if the range of R is the $\{0, 1\}$-subalgebra of $\mathbf{A}$. In crisp models, we have

$$\bar{e}(\Box\varphi, w) = \inf\{\bar{e}(\varphi, u) \; ; \; Rwu\}$$

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A **bilattice** is, roughly, an algebra with **two lattice orders**. The literature on bilattice-valued modal logics (Odintsov and Wansing, 2010; Rivieccio et al., 2017) considers languages where only one of the orders corresponds to a modal operator.

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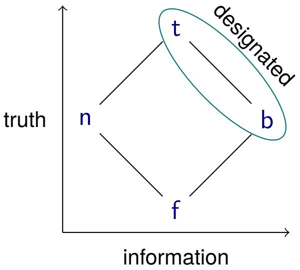
So what happens if we add a second one??

The Dunn–Belnap bilattice

Dunn (1966), Belnap (1977a, 1977b)



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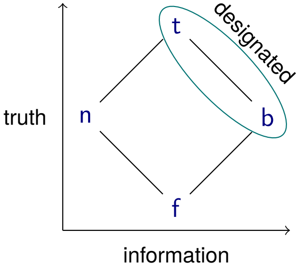

$$e(\varphi \wedge \psi) = \inf_{\leq_t}\{e(\varphi), e(\psi)\}$$

$$e(\varphi \vee \psi) = \sup_{\leq_t}\{e(\varphi), e(\psi)\}$$

$$e(\neg\varphi) = \begin{cases} \mathsf{t} & \text{if } e(\varphi) = \mathsf{f} \\ \mathsf{f} & \text{if } e(\varphi) = \mathsf{t} \\ e(\varphi) & \text{otherwise} \end{cases}$$

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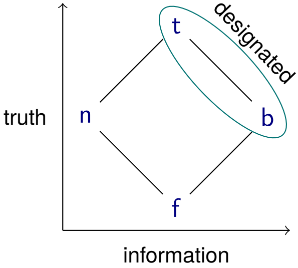

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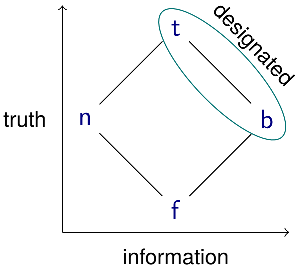
Arieli and Avron (1996), BL

$$e(\varphi \supset \psi) = \begin{cases} \mathsf{t} & \text{if } e(\varphi, w) \notin D \\ e(\psi) & \text{otherwise} \end{cases}$$

$$e(\varphi \supset \psi) \in D \text{ iff } (e(\varphi) \in D \implies e(\psi) \in D).$$

Some properties of DB

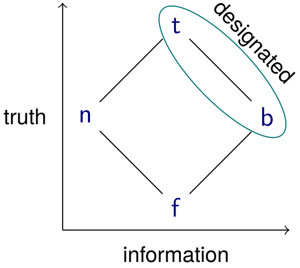


"Classical negation"
If $\sim\varphi := \varphi \supset$ f, then $e(\sim\varphi) \in D$ iff $e(\varphi) \notin D$; but, for example, not always $e(\sim\sim\varphi) = e(\varphi)$.

Expressing truth values $e(\varphi) = \begin{cases} \text{t} & \text{iff } \{\varphi, \sim\neg\varphi\} \in D \\ \text{f} & \text{iff } \{\sim\varphi, \neg\varphi\} \in D \\ \text{b} & \text{iff } \{\varphi, \neg\varphi\} \in D \\ \text{n} & \text{iff } \{\sim\varphi, \sim\neg\varphi\} \in D \end{cases}$

Filters
Both D and $\{x\ ;\ \sim\neg x \in D\}$ are prime filters wrt the truth order;
Both D and $\{x\ ;\ \neg x \in D\}$ is a prime filter wrt the info order

A modal Dunn–Belnap logic



Odintsov and Wansing (2010), BK

Language $\{\wedge, \vee, \neg, \supset, \mathsf{f}, \Box\}$,

$\mathbf{DB}$-valued crisp Kripke models; and

$e(\Box\varphi, w) = \inf_{\leq_t}\{e(\varphi, w')\ ;\ Rww'\}$

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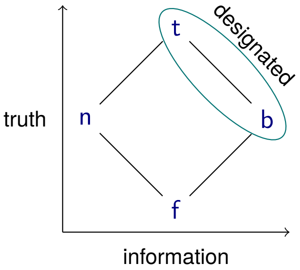

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Think of the states in a $\mathbf{DB}$-valued crisp model as possibly incomplete and inconsistent bodies of information within a network (graph). For example, agents in a social network, interconnected databases etc. A modal logic over such models expresses properties of and represents reasoning about such “information networks”.

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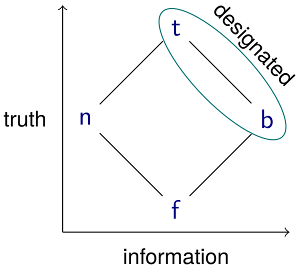

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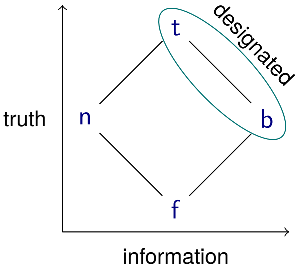

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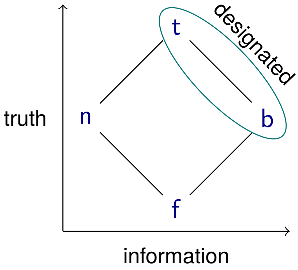
The story invites to consider an **information-order-based** modality as well!

Outline

The information box – motivation



BBK extends the language of BK by a new modality $\Box_i$ with the semantic clause

$$e(\Box_i \varphi, w) = \inf_{\leq_i} \{e(\varphi, w') \;;\; Rww'\}$$

Sources. Graphs represent "sources of information"; the value of $\Box p$ is the value that can be assigned to p after considering all the sources (i.e. the info on which all the sources agree).

Supervaluations. Graphs represent possibly incomplete or inconsistent valuations; $\Box p$ is the "supervalue" of p, i.e. the "least" value on which all the accessible "supervaluations" agree (cf. $p \supset \Box p$ and $\neg p \supset \Box \neg p$).

Some properties of BBK

$\Box\varphi \supset\subset \Box_i\varphi$ is valid, but $\neg\Box\varphi \supset\subset \neg\Box_i\varphi$ is not.

In fact, $\neg\Box_i\varphi \supset\subset \Box_i\neg\varphi$ is valid.

$\dfrac{\bigwedge\Gamma \supset \varphi}{\bigwedge\Box_i\Gamma \supset \Box_i\varphi}$ preserves validity.

Note: If n is added to the language, then the information modality is definable

$$\Box_i\varphi := (\mathsf{n} \wedge \neg\Box\neg\varphi) \vee \Box\varphi$$

Outline

The axiom system $\mathcal{BBK}$

Implication axioms

$\varphi \supset (\psi \supset \varphi)$

$(\varphi \supset (\psi \supset \chi)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \chi))$

$((\varphi \supset \psi) \supset \varphi) \supset \varphi$

Lattice axioms

$(\varphi \wedge \psi) \supset \varphi$ and $(\varphi \wedge \psi) \supset \psi$

$\varphi \supset (\varphi \vee \psi)$ and $\psi \supset (\varphi \vee \psi)$

$\varphi \supset (\psi \supset \varphi \wedge \psi)$

$(\varphi \supset \chi) \supset ((\psi \supset \chi) \supset (\varphi \vee \psi \supset \chi))$

$\mathsf{f} \supset \varphi$

Negation axioms

$\varphi \supset\subset \neg\neg\varphi$

$\varphi \supset \neg\mathsf{f}$

$\neg(\varphi \wedge \psi) \supset\subset (\neg\varphi \vee \neg\psi)$

$\neg(\varphi \vee \psi) \supset\subset (\neg\varphi \wedge \neg\psi)$

$\neg(\varphi \supset \psi) \supset\subset (\varphi \wedge \neg\psi)$

Modal "filter" axioms

$\Box{\sim}\neg\varphi \supset\subset {\sim}\neg\Box\varphi$

$\Box_i\neg\varphi \supset\subset \neg\Box_i\varphi$

Normality rules

$$\frac{\bigwedge \Gamma \supset \varphi}{\bigwedge \Box\Gamma \supset \Box\varphi}$$

$$\frac{\bigwedge \Gamma \supset \varphi}{\bigwedge \Box_i\Gamma \supset \Box_i\varphi} \quad \Gamma \subseteq_\omega Fm$$

Modus ponens

$$\frac{\varphi \quad \varphi \supset \psi}{\psi}$$

Completeness (Prime theories and extension)

A **nontrivial prime theory** is any set of formulas Γ such that

- $\Gamma \in \varphi$ iff $\Gamma \vdash \varphi$ ($\Gamma \vdash \varphi \ := \ \Gamma' \subseteq_\omega \Gamma$, provable $\bigwedge \Gamma' \supset \varphi$)
- $\Gamma \neq Fm$
- $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$

A pair of arbitrary sets of formulas $\langle \Gamma, \Delta \rangle$ is an **independent pair** iff there are no finite $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ where

$$\vdash \bigwedge \Gamma' \supset \bigvee \Delta'.$$

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Lemma 1 (Extension Lemma)

Let $\langle \Gamma, \Delta \rangle$ be an independent pair. Then there is a nontrivial prime theory Σ such that $\Gamma \subseteq \Sigma$ and $\Sigma \cap \Delta = \emptyset$.

Proof. See (Restall, 2000), ch. 5.2. ($\vdash$ is "pair extension acceptable".) $\square$

Completeness (Canonical model)

Let $\bar{\Gamma} = \{\varphi\ ;\ \Box\varphi \in \Gamma\}$. The **canonical model** is $M_c = \langle W_c, R_c, e_c \rangle$ defined as follows. W_c is the set of all nontrivial prime theories; $R_c\Gamma\Sigma$ iff $\bar{\Gamma} \subseteq \Sigma$ and

$$e_c(\varphi, \Gamma) = \begin{cases} \mathsf{b} & \text{if } \{\varphi, \neg\varphi\} \subseteq \Gamma \\ \mathsf{t} & \text{if } \{\varphi, {\sim}\neg\varphi\} \subseteq \Gamma \\ \mathsf{f} & \text{if } \{{\sim}\varphi, \neg\varphi\} \subseteq \Gamma \\ \mathsf{n} & \text{if } \{{\sim}\varphi, {\sim}\neg\varphi\} \subseteq \Gamma \end{cases}$$

Note that $\varphi \notin \Gamma$ iff ${\sim}\varphi \in \Gamma$ and $e_c(\varphi, \Gamma) \in D$ iff $\varphi \in \Gamma$.

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Lemma 2 (Witness Lemma)

In M_c, $\Box\varphi \in \Gamma \iff (\forall\Sigma)(R\Gamma\Sigma \implies \varphi \in \Sigma)$ and the same for $\Box_i$.

Proof. Def. of M_c and the Extension Lemma 1 (uses normality of $\Box$). $\Box$

Canonical Filter Lemma

Lemma 3

Let $X = \{e_c(\varphi, \Sigma)\ ;\ R_c\Gamma\Sigma\}$*. Then (*$D^f = \{x\ ;\ f(x) \in D\}$*)*

1. $\inf_o X \in D$ *iff* $e_c(\Box\varphi, \Gamma) \in D$ *for* $o \in \{t, i\}$

2. $\inf_t X \in D^{\sim\neg}$ *iff* $e_c(\Box\varphi, \Gamma) \in D^{\sim\neg}$

3. $\inf_i X \in D^{\neg}$ *iff* $e_c(\Box_i\varphi, \Gamma) \in D^{\neg}$

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3. $\inf_i X \in D^{\neg}$ *iff* $e_c(\Box_i\varphi, \Gamma) \in D^{\neg}$

Proof.

$\inf_i X \in D(\neg) \iff X \subseteq D^{\neg}$	Filter properties
$\iff \neg\varphi \in \Sigma$ for $R_c\Gamma\Sigma$	def. M_c
$\iff \Box\neg\varphi \in \Gamma$	Witness Lemma
$\iff \neg\Box\varphi \in \Gamma$	Filter axiom
$\iff e_c(\Box\varphi, \Gamma) \in D^{\neg}$	def. M_c

□

Completeness

Theorem 4

M_c is a four-valued Kripke model.

Proof. It is sufficient to show that $e_c(\Box\varphi, \Gamma) = \inf_t\{e_c(\varphi, \Sigma) ; R_c\Gamma\Sigma\}$ and that $e_c(\Box_i\varphi, \Gamma) = \inf_i\{e_c(\varphi, \Sigma) ; R_c\Gamma\Sigma\}$.

- the Canonical Filter Lemma 3
- every truth value $x \in \mathbf{DB}$ is "expressible" by means of $D, D^{\neg}$ (e.g. $x = \mathsf{t}$ iff $x \in D$ and $x \notin D^{\neg}$) and by means of $D, D^{\sim\neg}$ (e.g. $x = \mathsf{t}$ iff $x \in D$ and $x \in D^{\sim\neg}$)

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$\square$

Theorem 5

BBK $= Thm(\mathcal{BBK})$.

Generalizing the completeness argument

Assume that we have a matrix $\langle \mathbf{A}, D \rangle$ such that $\mathsf{f} \not\in D$ and $\supset^{\mathbf{A}}$ is an D-**implication** in the sense that $x \supset^{\mathbf{A}} y \in D$ iff $(x \in D$ only if $y \in D)$. Let us assume that D is a **complete prime filter**.

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Lemma 6 (Prime Extension Property)

If $\mathcal{H}$ is complete wrt $\vdash_{\mathbf{A}}$ (defined over non-modal formulas), then every independent $\vdash_{\mathcal{H}}$-pair $\langle \Gamma, \Delta \rangle$ is extendible to a non-trivial prime theory Σ s.t. $\Gamma \subseteq \Sigma$ and $\Sigma \cap \Delta$ is empty.

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Assume that every $x \in \mathbf{A}$ is **expressible** by a **unique** set of unary operators $E(x) \subseteq U$ in the sense that, for every unary operator $f \in U$ (definable in the language) including identity and for all $y \in \mathbf{A}$

$$x = y \text{ iff } f(y) \in D \iff f \in E(x)$$

Assume that D^f is a complete prime filter for all $f \in U$.

Generalizing the completeness argument

Theorem 7

$\mathcal{H}$ plus the normality rule and the filter axioms $f(\Box\varphi) \supset\subset \Box f(\varphi)$ is complete wrt the class of $\mathbf{A}$-valued crisp Kripke models.

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$$
\begin{array}{llr}
\inf X \in D^f & \iff X \subseteq D^f & \text{Filter properties} \\
& \iff f(\varphi) \in \Sigma \text{ for } R_c\Gamma\Sigma & \text{def. } M_c \text{ / } \mathcal{H} \text{ is } \mathbf{A}\text{-compl.} \\
& \iff \Box f(\varphi) \in \Gamma & \text{Witness Lemma} \\
& \iff f(\Box\varphi) \in \Gamma & \text{Filter axiom} \\
& \iff e_c(\Box\varphi, \Gamma) \in D^f & \text{def. } M_c
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$\inf X \in D^f$	$\iff$	$X \subseteq D^f$	Filter properties
	$\iff$	$f(\varphi) \in \Sigma$ for $R_c\Gamma\Sigma$	def. M_c / $\mathcal{H}$ is $\mathbf{A}$-compl.
	$\iff$	$\Box f(\varphi) \in \Gamma$	Witness Lemma
	$\iff$	$f(\Box\varphi) \in \Gamma$	Filter axiom
	$\iff$	$e_c(\Box\varphi, \Gamma) \in D^f$	def. M_c

The Modal Truth Lemma holds because U expresses $\mathbf{A}$.

□

Conclusion

- From the viewpoint of informal interpretation, it makes sense to study **bimodal** bilattice-valued logic with a truth-order-based modality and an **information-order-based modality** (more work on applications and expressivity later)
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Thank you!

References

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Pair extension acceptability

$\varphi \vdash \varphi$

$\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$

If $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$

$\varphi \vdash \varphi \vee \psi$ and $\psi \vdash \varphi \vee \psi$

If $\varphi \vdash \chi$ and $\psi \vdash \chi$, then $\varphi \vee \psi \vdash \chi$

$\varphi \wedge (\psi_1 \vee \psi_2) \vdash (\varphi \wedge \psi_1) \vee (\varphi \wedge \psi_2)$

If $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$

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