Finite model properties for many-valued modal logics.

Willem Conradie [1] Wilmari Morton [1] Claudette Robinson [2]

[1]Department of Pure and Applied Mathematics,
University of Johannesburg.

[2]Department of Computer Science, University of the Witwatersrand.

Thursday, 29 June 2017
Prague, Czech Republic

Residuated lattices

An algebra $\mathbf{A} = (A, \wedge, \vee, \circ, \rightarrow, 0, 1)$ is a residuated lattice if, and only if:

- the reduct $(A, \wedge, \vee, 0, 1)$ is a bounded lattice with top element 1 and bottom element 0 (we will denote its order by $\leq$),
- the reduct $(A, \circ, 1)$ is a commutative monoid with identity 1, and
- the fusion operation (also called strong conjunction) is residuated, with $\rightarrow$ being its right residual; that is, for all $a, b, c \in A$,

$$a \circ b \leq c \Longleftrightarrow a \leq b \rightarrow c.$$

Residuated lattices

An algebra $\mathbf{A} = (A, \wedge, \vee, \circ, \rightarrow, 0, 1)$ is a residuated lattice if, and only if:

- the reduct $(A, \wedge, \vee, 0, 1)$ is a bounded lattice with top element 1 and bottom element 0 (we will denote its order by $\leq$),
- the reduct $(A, \circ, 1)$ is a commutative monoid with identity 1, and
- the fusion operation (also called strong conjunction) is residuated, with $\rightarrow$ being its right residual; that is, for all $a, b, c \in A$,

$$a \circ b \leq c \Longleftrightarrow a \leq b \rightarrow c.$$

Residuated lattices

An algebra $\mathbf{A} = (A, \wedge, \vee, \circ, \rightarrow, 0, 1)$ is a residuated lattice if, and only if:

- the reduct $(A, \wedge, \vee, 0, 1)$ is a bounded lattice with top element 1 and bottom element 0 (we will denote its order by $\leq$),
- the reduct $(A, \circ, 1)$ is a commutative monoid with identity 1, and
- the fusion operation (also called strong conjunction) is residuated, with $\rightarrow$ being its right residual; that is, for all $a, b, c \in A$,

$$a \circ b \leq c \iff a \leq b \rightarrow c.$$

Residuated lattices

An algebra $\mathbf{A} = (A, \wedge, \vee, \circ, \to, 0, 1)$ is a residuated lattice if, and only if:

- the reduct $(A, \wedge, \vee, 0, 1)$ is a bounded lattice with top element 1 and bottom element 0 (we will denote its order by $\leq$),
- the reduct $(A, \circ, 1)$ is a commutative monoid with identity 1, and
- the fusion operation (also called strong conjunction) is residuated, with $\to$ being its right residual; that is, for all $a, b, c \in A$,

$$a \circ b \leq c \iff a \leq b \to c.$$

Residuated lattices

An algebra $\mathbf{A} = (A, \wedge, \vee, \circ, \rightarrow, 0, 1)$ is a residuated lattice if, and only if:

- the reduct $(A, \wedge, \vee, 0, 1)$ is a bounded lattice with top element 1 and bottom element 0 (we will denote its order by $\leq$),
- the reduct $(A, \circ, 1)$ is a commutative monoid with identity 1, and
- the fusion operation (also called strong conjunction) is residuated, with $\rightarrow$ being its right residual; that is, for all $a, b, c \in A$,

$$a \circ b \leq c \iff a \leq b \rightarrow c.$$

We will require completeness in the residuated lattices.

The languages

Fix a residuated lattice $\mathbf{A} = (A, \wedge, \vee, \circ, \rightarrow, 0, 1)$ and a countably infinite set of propositional variables Φ.

The languages

The set of formulas $\mathrm{Fm}(\mathbf{A})$ of the many-valued propositional language over Φ and $\mathbf{A}$ is given by the following recursive definition:

$$\varphi := \mathbf{t} \mid p \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \psi_1 \circ \psi_2 \mid \psi_1 \to \psi_2.$$

Here $t \in A$ and $p \in \Phi$. We will refer to $\mathbf{A}$ as the truth-value space and the elements $\mathbf{t}$ as truth-value constants.

The languages

The set of formulas $\mathrm{Fm}(\mathbf{A})$ of the many-valued propositional language over Φ and $\mathbf{A}$ is given by the following recursive definition:

$$\varphi := \mathbf{t} \mid p \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \psi_1 \circ \psi_2 \mid \psi_1 \to \psi_2.$$

Here $t \in A$ and $p \in \Phi$. We will refer to $\mathbf{A}$ as the truth-value space and the elements $\mathbf{t}$ as truth-value constants.

We define the set $\mathrm{Fm}_{\Diamond,\Box}(\mathbf{A})$ of formulas of the basic many-valued modal language over Φ and $\mathbf{A}$ as follows:

$$\varphi := \mathbf{t} \mid p \mid \psi_1 \vee \psi_2 \mid \psi_1 \wedge \psi_2 \mid \psi_1 \to \psi_2 \mid \Diamond\psi \mid \Box\psi.$$

The languages

The set of formulas $\mathrm{Fm}(\mathbf{A})$ of the many-valued propositional language over Φ and $\mathbf{A}$ is given by the following recursive definition:

$$\varphi := \mathbf{t} \mid p \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \psi_1 \circ \psi_2 \mid \psi_1 \to \psi_2.$$

Here $t \in A$ and $p \in \Phi$. We will refer to $\mathbf{A}$ as the truth-value space and the elements $\mathbf{t}$ as truth-value constants.

We define the set $\mathrm{Fm}_{\Diamond,\Box}(\mathbf{A})$ of formulas of the basic many-valued modal language over Φ and $\mathbf{A}$ as follows:

$$\varphi := \mathbf{t} \mid p \mid \psi_1 \vee \psi_2 \mid \psi_1 \wedge \psi_2 \mid \psi_1 \to \psi_2 \mid \Diamond\psi \mid \Box\psi.$$

The subsets of formulas $\mathrm{Fm}_{\Diamond}(\mathbf{A})$ and $\mathrm{Fm}_{\Box}(\mathbf{A})$ are then defined in the obvious way.

Frames and models

Definition

A many-valued (Kripke) frame over **A** (**A**-frame) for the basic many-valued modal language is a triple $\mathfrak{F} = (W, D, B)$ with a nonempty universe W and many-valued accessibility relations such that $D : W \times W \to A$ and $B : W \times W \to A$.

Frames and models

Definition

A many-valued (Kripke) frame over **A** (**A**-frame) for the basic many-valued modal language is a triple $\mathfrak{F} = (W, D, B)$ with a nonempty universe W and many-valued accessibility relations such that $D : W \times W \to A$ and $B : W \times W \to A$.

Note that our two modalities will not be interdefinable, in general.

Frames and models

Definition

A many-valued (Kripke) frame over $\mathbf{A}$ ($\mathbf{A}$-frame) for the basic many-valued modal language is a triple $\mathfrak{F} = (W, D, B)$ with a nonempty universe W and many-valued accessibility relations such that $D : W \times W \to A$ and $B : W \times W \to A$.

Note that our two modalities will not be interdefinable, in general.

Definition

A many-valued (Kripke) model over $\mathbf{A}$ ($\mathbf{A}$-model) for the basic many-valued modal language is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is an $\mathbf{A}$-frame and V is a map (called a valuation) such that V: $\mathrm{PROP} \times W \to A$ and $V(\mathbf{t}, w) = t$ for all $w \in W$ and $t \in A$.

Semantics

The valuation can be extended to all formulas as follows:

$$
\begin{aligned}
V(\psi \wedge \varphi, w) &= V(\psi, w) \wedge V(\varphi, w), \\
V(\psi \vee \varphi, w) &= V(\psi, w) \vee V(\varphi, w), \\
V(\psi \circ \varphi, w) &= V(\psi, w) \circ V(\varphi, w), \\
V(\psi \rightarrow \varphi, w) &= V(\psi, w) \rightarrow V(\varphi, w), \\
V(\Diamond\psi, w) &= \bigvee\{D(w, v) \circ V(\psi, v) \mid v \in W\} \text{ and} \\
V(\Box\psi, w) &= \bigwedge\{B(w, v) \rightarrow V(\psi, v) \mid v \in W\}.
\end{aligned}
$$

Semantics

The valuation can be extended to all formulas as follows:

$$
\begin{aligned}
V(\psi \wedge \varphi, w) &= V(\psi, w) \wedge V(\varphi, w),\\
V(\psi \vee \varphi, w) &= V(\psi, w) \vee V(\varphi, w),\\
V(\psi \circ \varphi, w) &= V(\psi, w) \circ V(\varphi, w),\\
V(\psi \rightarrow \varphi, w) &= V(\psi, w) \rightarrow V(\varphi, w),\\
V(\Diamond\psi, w) &= \bigvee\{D(w, v) \circ V(\psi, v) \mid v \in W\} \text{ and}\\
V(\Box\psi, w) &= \bigwedge\{B(w, v) \rightarrow V(\psi, v) \mid v \in W\}.
\end{aligned}
$$

We know $V(\Diamond\psi, w)$ and $V(\Box\psi, w)$ will always exist in $\mathbf{A}$ because it is complete.

Degrees of truth and validiy

Let $a \in A$.

Degrees of truth and validiy

We say that φ is a-true in an $\mathbf{A}$-model $\mathfrak{M}$ at $w \in W$ (denoted $\mathfrak{M}, w \Vdash^a \varphi$), if $V(\varphi, w) \geq a$.

Degrees of truth and validiy

We say that φ is a-true in an **A**-model $\mathfrak{M}$ at $w \in W$ (denoted $\mathfrak{M}, w \Vdash^a \varphi$), if $V(\varphi, w) \geq a$.

A formula φ is a-true in an **A**-model $\mathfrak{M}$ (denoted $\mathfrak{M} \Vdash^a \varphi$), if $V(\varphi, w) \geq a$ for all $w \in W$.

Degrees of truth and validiy

We say that φ is a-true in an **A**-model $\mathfrak{M}$ at $w \in W$ (denoted $\mathfrak{M}, w \Vdash^a \varphi$), if $V(\varphi, w) \geq a$.

A formula φ is a-true in an **A**-model $\mathfrak{M}$ (denoted $\mathfrak{M} \Vdash^a \varphi$), if $V(\varphi, w) \geq a$ for all $w \in W$.

A formula φ is a-valid in an **A**-frame $\mathfrak{F}$ at $w \in W$ (denoted $\mathfrak{F}, w \Vdash^a \varphi$), if $V(\varphi, w) \geq a$ for all valuations V on $\mathfrak{F}$.

Degrees of truth and validiy

We say that φ is a-true in an **A**-model $\mathfrak{M}$ at $w \in W$ (denoted $\mathfrak{M}, w \Vdash^a \varphi$), if $V(\varphi, w) \geq a$.

A formula φ is a-true in an **A**-model $\mathfrak{M}$ (denoted $\mathfrak{M} \Vdash^a \varphi$), if $V(\varphi, w) \geq a$ for all $w \in W$.

A formula φ is a-valid in an **A**-frame $\mathfrak{F}$ at $w \in W$ (denoted $\mathfrak{F}, w \Vdash^a \varphi$), if $V(\varphi, w) \geq a$ for all valuations V on $\mathfrak{F}$.

Finally, we say φ is a-valid in an **A**-frame $\mathfrak{F}$ (denoted $\mathfrak{F} \Vdash^a \varphi$), if $\mathfrak{F}, w \Vdash^a \varphi$ for all $w \in W$.

The non-modal logic of a residuated lattice

The non-modal logic of the many-valued propositional language over Φ and $\mathbf{A}$ will be denoted by $\Lambda(\mathbf{A})$ and is obtained by setting for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}(\mathbf{A})$,

$$\Gamma \Vdash_{\mathbf{A}} \varphi \text{ iff } \forall h \in \mathrm{HOM}(\mathbf{Fm}(\mathbf{A}), \mathbf{A}),\ \text{if } h[\Gamma] \subseteq \{1\},\ \text{then } h(\varphi) = 1.$$

The non-modal logic of a residuated lattice

The non-modal logic of the many-valued propositional language over Φ and $\mathbf{A}$ will be denoted by $\Lambda(\mathbf{A})$ and is obtained by setting for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}(\mathbf{A})$,

$$\Gamma \Vdash_{\mathbf{A}} \varphi \text{ iff } \forall h \in \mathrm{HOM}(\mathbf{Fm}(\mathbf{A}), \mathbf{A}),\ \text{if } h[\Gamma] \subseteq \{1\},\ \text{then } h(\varphi) = 1.$$

If $\mathbf{A}$ is finite, then $\Lambda(\mathbf{A})$ is finitary.

The non-modal logic of a residuated lattice

The non-modal logic of the many-valued propositional language over Φ and $\mathbf{A}$ will be denoted by $\Lambda(\mathbf{A})$ and is obtained by setting for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}(\mathbf{A})$,

$$\Gamma \Vdash_{\mathbf{A}} \varphi \text{ iff } \forall h \in \mathrm{HOM}(\mathbf{Fm}(\mathbf{A}), \mathbf{A}),\ \text{if } h[\Gamma] \subseteq \{1\},\ \text{then } h(\varphi) = 1.$$

If $\mathbf{A}$ is finite, then $\Lambda(\mathbf{A})$ is finitary.

Throughout this talk we will assume that we have a sound and complete axiomatization for $\Lambda(\mathbf{A})$.

The basic many-valued modal logic

In what follows, we restrict our attention to **A**-frames (W, D, B) where $D = B = R$.

The basic many-valued modal logic

Moreover, we will assume that $\mathbf{A}$ is a finite residuated lattice.

The basic many-valued modal logic

Let $\mathsf{K}(\mathbf{A})$ be a class of $\mathbf{A}$-frames. We will denote the set $\{\psi \in \mathsf{Fm}_{\Diamond,\Box}(\mathbf{A}) \mid \mathsf{K}(\mathbf{A}) \Vdash^1 \psi\}$ by $\Lambda_{\mathsf{K}(\mathbf{A})}$.

The basic many-valued modal logic

Let $\mathsf{K}(\mathbf{A})$ be a class of $\mathbf{A}$-frames. We will denote the set $\{\psi \in \mathsf{Fm}_{\Diamond,\Box}(\mathbf{A}) \mid \mathsf{K}(\mathbf{A}) \Vdash^1 \psi\}$ by $\Lambda_{\mathsf{K}(\mathbf{A})}$.

Let $\mathsf{Fr}(\mathbf{A})$ be the class of all $\mathbf{A}$-frames (W, R). The (local) many-valued modal logic $\Lambda(\ell, \mathsf{Fr}(\mathbf{A}))$ associated with $\mathbf{A}$ and the class of $\mathbf{A}$-frames $\mathsf{Fr}(\mathbf{A})$ is obtained by setting for all $\Gamma \cup \{\varphi\} \subseteq \mathsf{Fm}_{\Diamond,\Box}(\mathbf{A})$,

The basic many-valued modal logic

Let $\mathsf{K}(\mathbf{A})$ be a class of $\mathbf{A}$-frames. We will denote the set $\{\psi \in \mathrm{Fm}_{\Diamond,\Box}(\mathbf{A}) \mid \mathsf{K}(\mathbf{A}) \Vdash^{1} \psi\}$ by $\Lambda_{\mathsf{K}(\mathbf{A})}$.

Let $\mathrm{Fr}(\mathbf{A})$ be the class of all $\mathbf{A}$-frames (W, R). The (local) many-valued modal logic $\Lambda(\ell, \mathrm{Fr}(\mathbf{A}))$ associated with $\mathbf{A}$ and the class of $\mathbf{A}$-frames $\mathrm{Fr}(\mathbf{A})$ is obtained by setting for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\Diamond,\Box}(\mathbf{A})$,

$\Gamma \Vdash^{\ell}_{\mathrm{Fr}(\mathbf{A})} \varphi$ iff for every $\mathbf{A}$-model (W, R, V) over a frame (W, R) in $\mathrm{Fr}(\mathbf{A})$ and for every $w \in W$, it holds that if $V(\gamma, w) = 1$, for every $\gamma \in \Gamma$, then $V(\varphi, w) = 1$.

Many-valued modal logic sets

Definition

A many-valued modal logic set over $\mathbf{A}$ is any set $L(\mathbf{A})$ of formulas that contains all axioms for $\Lambda(\mathbf{A})$ and all axioms in the table on the next slide, and which is closed under the inference rules for $\Lambda(\mathbf{A})$ and the inference rules in the table.

The smallest many-valued modal logic set over $\mathbf{A}$ will be denoted by $\mathbf{K}(\mathbf{A})$.

Many-valued modal logic sets (cont.)

Axioms:

$\vdash \Box(p \wedge q) \to (\Box p \wedge \Box q)$ $\quad$ $\vdash \Box\mathbf{1}$

$\vdash \Box(\mathbf{a} \to p) \leftrightarrow (\mathbf{a} \to \Box p)$ $\quad$ $\vdash \Box(p \to \mathbf{a}) \leftrightarrow (\Diamond p \to \mathbf{a})$

Rules of inference:

$\vdash \varphi'$ when $\vdash \varphi$, where φ' is obtained from φ by substitution.

If $\vdash \varphi \to \psi$, then $\vdash \Box\varphi \to \Box\psi$ (called monotonicity).

Completeness of the basic many-valued modal logic when $\mathbf{A}$ *is finite*

Theorem

Let $\mathbf{A}$ *be a finite residuated lattice. Then:*

1. $\mathbf{K}(\mathbf{A}) = \Lambda_{\mathrm{Fr}(\mathbf{A})}$.
2. *The logic* $\Lambda(\ell, \mathrm{Fr}(\mathbf{A}))$ *is axiomatized by* $\mathbf{K}(\mathbf{A})$ *as a set of axioms and the rules for* $\Lambda(\mathbf{A})$.

Completeness of the basic many-valued modal logic when $\mathbf{A}$ is finite

Theorem

Let $\mathbf{A}$ *be a finite residuated lattice. Then:*

1. $\mathbf{K}(\mathbf{A}) = \Lambda_{\mathrm{Fr}(\mathbf{A})}$.
2. *The logic* $\Lambda(\ell, \mathrm{Fr}(\mathbf{A}))$ *is axiomatized by* $\mathbf{K}(\mathbf{A})$ *as a set of axioms and the rules for* $\Lambda(\mathbf{A})$.

Remark

Bou et. al. (2011) axiomatized the basic many-valued modal logic of all $\mathbf{A}$*-frames for the many-valued modal language containing only* $\Box$.

Some extensions of $\mathbf{K}(\mathbf{A})$

$$\mathbf{T}(\mathbf{A}) = \mathbf{K}(\mathbf{A}) \oplus \{p \to \Diamond p, \Box p \to p\}$$
$$\mathbf{D}(\mathbf{A}) = \mathbf{K}(\mathbf{A}) \oplus \{\Box p \to \Diamond p\}$$
$$\mathbf{B}(\mathbf{A}) = \mathbf{K}(\mathbf{A}) \oplus \{p \to \Box\Diamond p, \Diamond\Box p \to p\}$$
$$\mathbf{K4}(\mathbf{A}) = \mathbf{K}(\mathbf{A}) \oplus \{\Diamond\Diamond p \to \Diamond p, \Box p \to \Box\Box p\}$$
$$\mathbf{S4}(\mathbf{A}) = \mathbf{T}(\mathbf{A}) \oplus \{\Diamond\Diamond p \to \Diamond p, \Box p \to \Box\Box p\}$$

Equivalence classes

Fix a residuated lattice $\mathbf{A} = (A, \wedge, \vee, \circ, \rightarrow, 0, 1)$.

Equivalence classes

Recall that Σ is called subformula closed if, whenever $\varphi \in \Sigma$ and ψ is a subformula of φ, then $\psi \in \Sigma$.

Equivalence classes

Let $\mathfrak{M} = (W, D, B, V)$ be an $\mathbf{A}$-model and Σ a subformula closed set of formulas. Let $\leftrightsquigarrow^{\mathbf{A}}_{\Sigma}$ be the relation on W defined by:

$$\text{for } w, v \in W \text{ and } \varphi \in \Sigma,\ w \leftrightsquigarrow^{\mathbf{A}}_{\Sigma} v \text{ if, and only if,}$$
$$\mathfrak{M}, w \Vdash^{a} \varphi \iff \mathfrak{M}, v \Vdash^{a} \varphi \text{ for all } a \in \mathbf{A}.$$

Equivalence classes

Let $\mathfrak{M} = (W, D, B, V)$ be an $\mathbf{A}$-model and Σ a subformula closed set of formulas. Let $\leftrightsquigarrow^{\mathbf{A}}_{\Sigma}$ be the relation on W defined by:

$$\text{for } w, v \in W \text{ and } \varphi \in \Sigma,\ w \leftrightsquigarrow^{\mathbf{A}}_{\Sigma} v \text{ if, and only if,}$$
$$\mathfrak{M}, w \Vdash^{a} \varphi \iff \mathfrak{M}, v \Vdash^{a} \varphi \text{ for all } a \in \mathbf{A}.$$

The relation $\leftrightsquigarrow^{\mathbf{A}}_{\Sigma}$ is clearly an equivalence relation.

Equivalence classes

Let $\mathfrak{M} = (W, D, B, V)$ be an $\mathbf{A}$-model and Σ a subformula closed set of formulas. Let $\leftrightsquigarrow^{\mathbf{A}}_{\Sigma}$ be the relation on W defined by:

for $w, v \in W$ and $\varphi \in \Sigma$, $w \leftrightsquigarrow^{\mathbf{A}}_{\Sigma} v$ if, and only if,
$\mathfrak{M}, w \Vdash^{a} \varphi \iff \mathfrak{M}, v \Vdash^{a} \varphi$ for all $a \in \mathbf{A}$.

The relation $\leftrightsquigarrow^{\mathbf{A}}_{\Sigma}$ is clearly an equivalence relation.

We denote the equivalence class of $w \in W$ by $[w]$.

Equivalence classes

Let $\mathfrak{M} = (W, D, B, V)$ be an **A**-model and Σ a subformula closed set of formulas. Let $\leftrightsquigarrow^{\mathbf{A}}_{\Sigma}$ be the relation on W defined by:

$$\text{for } w, v \in W \text{ and } \varphi \in \Sigma,\ w \leftrightsquigarrow^{\mathbf{A}}_{\Sigma} v \text{ if, and only if,}$$
$$\mathfrak{M}, w \Vdash^{a} \varphi \iff \mathfrak{M}, v \Vdash^{a} \varphi \text{ for all } a \in \mathbf{A}.$$

The relation $\leftrightsquigarrow^{\mathbf{A}}_{\Sigma}$ is clearly an equivalence relation.

We denote the equivalence class of $w \in W$ by $[w]$.

Set $W^{\mathbf{A}}_{\Sigma} = \{[w] \mid w \in W\}$.

Filtrations for many-valued Kripke models

Definition

Let $a, b \in A$. Suppose $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = \left(W_f^{\mathbf{A}}, D_f^{(\Sigma,\mathbf{A})}, B_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}}\right)$ is any model such that:

Filtrations for many-valued Kripke models

Definition

Let $a, b \in A$. Suppose $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = \left(W_f^{\mathbf{A}}, D_f^{(\Sigma,\mathbf{A})}, B_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}} \right)$ is any model such that:

(W) $W_f^{\mathbf{A}} = W_\Sigma^{\mathbf{A}}$.

Filtrations for many-valued Kripke models

Definition

Let $a, b \in A$. Suppose $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = \left(W_f^{\mathbf{A}}, D_f^{(\Sigma,\mathbf{A})}, B_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}}\right)$ is any model such that:

($R1$) If $D(w, v) \geq a$, then $D_f^{(\Sigma,\mathbf{A})}([w], [v]) \geq a$.

($R2$) If $D_f^{(\Sigma,\mathbf{A})}([w], [v]) \geq a$, then, for every $\Diamond\varphi \in \Sigma$, we have $\mathfrak{M}, w \Vdash^{a \circ b} \Diamond\varphi$ whenever $\mathfrak{M}, v \Vdash^{b} \varphi$.

Filtrations for many-valued Kripke models

Definition

Let $a, b \in A$. Suppose $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = \left(W_f^{\mathbf{A}}, D_f^{(\Sigma,\mathbf{A})}, B_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}}\right)$ is any model such that:

($R3$) If $B(w, v) \geq a$, then $B_f^{(\Sigma,\mathbf{A})}([w], [v]) \geq a$.

($R4$) If $B_f^{(\Sigma,\mathbf{A})}([w], [v]) \geq a$, then, for every $\Box\varphi \in \Sigma$, we have $\mathfrak{M}, v \Vdash^{a \circ b} \varphi$ whenever $\mathfrak{M}, w \Vdash^{b} \Box\varphi$.

Filtrations for many-valued Kripke models

Definition

Let $a, b \in A$. Suppose $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = \left(W_f^{\mathbf{A}}, D_f^{(\Sigma,\mathbf{A})}, B_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}}\right)$ is any model such that:

(V) $V_f^{\mathbf{A}}([w], p) = V(w, p)$ for all $p \in \Sigma$.

Filtrations for many-valued Kripke models

Definition

Let $a, b \in A$. Suppose $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = \left(W_f^{\mathbf{A}}, D_f^{(\Sigma,\mathbf{A})}, B_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}}\right)$ is any model such that:

(W) $W_f^{\mathbf{A}} = W_\Sigma^{\mathbf{A}}$.

($R1$) If $D(w,v) \geq a$, then $D_f^{(\Sigma,\mathbf{A})}([w],[v]) \geq a$.

($R2$) If $D_f^{(\Sigma,\mathbf{A})}([w],[v]) \geq a$, then, for every $\Diamond\varphi \in \Sigma$, we have $\mathfrak{M}, w \Vdash^{a \circ b} \Diamond\varphi$ whenever $\mathfrak{M}, v \Vdash^{b} \varphi$.

($R3$) If $B(w,v) \geq a$, then $B_f^{(\Sigma,\mathbf{A})}([w],[v]) \geq a$.

($R4$) If $B_f^{(\Sigma,\mathbf{A})}([w],[v]) \geq a$, then, for every $\Box\varphi \in \Sigma$, we have $\mathfrak{M}, v \Vdash^{a \circ b} \varphi$ whenever $\mathfrak{M}, w \Vdash^{b} \Box\varphi$.

(V) $V_f^{\mathbf{A}}([w],p) = V(w,p)$ for all $p \in \Sigma$.

Then $\mathfrak{M}_f^{(\Sigma,\mathbf{A})}$ is called a filtration of $\mathfrak{M}$ through Σ over $\mathbf{A}$.

The many-valued filtration theorem

Theorem

Let $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = (W_\Sigma^{\mathbf{A}}, D_f^{(\Sigma,\mathbf{A})}, B_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}})$ *be a filtration of* $\mathfrak{M}$ *through a subformula closed set* Σ *over* $\mathbf{A}$*. Then, for all formulas* $\varphi \in \Sigma$*, all states* w *in* $\mathfrak{M}$ *and any truth value* $a \neq 0$ *in* $\mathbf{A}$*, we have that*

$$\mathfrak{M}, w \Vdash^a \varphi \iff \mathfrak{M}_f^{(\Sigma,\mathbf{A})}, [w] \Vdash^a \varphi.$$

Moreover, if A *and* Σ *are both finite, then so is* $\mathfrak{M}_f^{(\Sigma,\mathbf{A})}$*.*

The smallest filtration

$$D_s^{(\Sigma,\mathbf{A})}([w],[v]) = \bigvee \{D(w',v') \mid w' \in [w], v' \in [v]\}$$

$$B_s^{(\Sigma,\mathbf{A})}([w],[v]) = \bigvee \{B(w',v') \mid w' \in [w], v' \in [v]\}$$

The smallest filtration

$$D_s^{(\Sigma,\mathbf{A})}([w],[v]) = \bigvee \{D(w',v') \mid w' \in [w], v' \in [v]\}$$

$$B_s^{(\Sigma,\mathbf{A})}([w],[v]) = \bigvee \{B(w',v') \mid w' \in [w], v' \in [v]\}$$

Proposition

Let Σ be a subformula closed set of formulas, and let $\mathfrak{M}_s^{(\Sigma,\mathbf{A})} = (W_\Sigma^{\mathbf{A}}, \mathrm{D}_s^{(\Sigma,\mathbf{A})}, \mathrm{B}_s^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}})$, where $D_s^{(\Sigma,\mathbf{A})}$ and $B_s^{(\Sigma,\mathbf{A})}$ are obtained as described above. Then $\mathfrak{M}_s^{(\Sigma,\mathbf{A})}$ is a filtration of $\mathfrak{M}$ through Σ over $\mathbf{A}$. Moreover, if $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = (W_\Sigma^{\mathbf{A}}, D_f^{(\Sigma,\mathbf{A})}, B_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}})$ is any filtration of $\mathfrak{M}$ through Σ over $\mathbf{A}$, then for all $[w],[v] \in W_\Sigma^{\mathbf{A}}$, we have that $D_s^{(\Sigma,\mathbf{A})}([w],[v]) \leq D_f^{(\Sigma,\mathbf{A})}([w],[v])$ and $B_s^{(\Sigma,\mathbf{A})}([w],[v]) \leq B_f^{(\Sigma,\mathbf{A})}([w],[v])$.

The largest filtration

$$D_{\ell}^{(\Sigma,\mathbf{A})}([w],[v]) = \bigwedge \{V(\varphi, v) \to V(\Diamond\varphi, w) \mid \Diamond\varphi \in \Sigma\}$$

$$B_{\ell}^{(\Sigma,\mathbf{A})}([w],[v]) = \bigwedge \{V(\Box\varphi, w) \to V(\varphi, v) \mid \Box\varphi \in \Sigma\}$$

The largest filtration

$$D_\ell^{(\Sigma,\mathbf{A})}([w],[v]) = \bigwedge \{ V(\varphi, v) \to V(\Diamond\varphi, w) \mid \Diamond\varphi \in \Sigma \}$$

$$B_\ell^{(\Sigma,\mathbf{A})}([w],[v]) = \bigwedge \{ V(\Box\varphi, w) \to V(\varphi, v) \mid \Box\varphi \in \Sigma \}$$

Proposition

Let Σ be a subformula closed set of formulas, and let $\mathfrak{M}_\ell^{(\Sigma,\mathbf{A})} = (W_\Sigma^{\mathbf{A}}, D_\ell^{(\Sigma,\mathbf{A})}, B_\ell^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}})$ where $D_\ell^{(\Sigma,\mathbf{A})}$ and $B_\ell^{(\Sigma,\mathbf{A})}$ are obtained as described above. Then $\mathfrak{M}_\ell^{(\Sigma,\mathbf{A})}$ is a filtration of $\mathfrak{M}$ through Σ over $\mathbf{A}$. Moreover, if $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = (W_\Sigma^{\mathbf{A}}, D_f^{(\Sigma,\mathbf{A})}, B_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}})$ is any filtration of $\mathfrak{M}$ through Σ over $\mathbf{A}$, then for all $[w],[v] \in W_\Sigma^{\mathbf{A}}$, we have that $D_\ell^{(\Sigma,\mathbf{A})}([w],[v]) \geq D_f^{(\Sigma,\mathbf{A})}([w],[v])$ and $B_\ell^{\mathbf{A}}([w],[v]) \geq B_f^{\mathbf{A}}([w],[v])$.

Many-valued frame conditions

An $\mathbf{A}$-frame $\mathfrak{F} = (W, D, B)$ is *a*-reflexive if $D(w, w) \geq a$ and $B(w, w) \geq a$ for all $w \in W$.

Many-valued frame conditions

An $\mathbf{A}$-frame $\mathfrak{F} = (W, D, B)$ is *a*-serial if $\bigvee_{v \in W} D(u, v) \geq a$ and $\bigvee_{v \in W} B(u, v) \geq a$ for all $u \in W$.

Many-valued frame conditions

An **A**-frame $\mathfrak{F} = (W, D, B)$ is a-symmetric if, for all $u, v \in W$,

$$a \leq D(u, v) \to D(v, u) \text{ and } a \leq B(u, v) \to B(v, u).$$

Many-valued frame conditions

An **A**-frame $\mathfrak{F} = (W, D, B)$ is *a*-transitive if, for all $u, v, w \in W$,

$$a \leq D(u, v) \circ D(v, w) \to D(u, w)$$

and

$$a \leq B(u, v) \circ B(v, w) \to B(u, w).$$

Frame conditions preserved by filtrations

Proposition

Let Σ be a subformula closed set of formulas, and let $\mathfrak{M}_f^{(\Sigma,\mathbf{A})} = (W_\Sigma^{\mathbf{A}}, \mathrm{D}_f^{(\Sigma,\mathbf{A})}, \mathrm{B}_f^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}})$ be a filtration of $\mathfrak{M} = (W, D, B, V)$ through Σ over $\mathbf{A}$. Then we have the following for $R \in \{D, B\}$:

1. *If R is a-reflexive, then so is $R_f^{(\Sigma,\mathbf{A})}$.*
2. *If R is a-serial, then so is $R_f^{(\Sigma,\mathbf{A})}$.*
3. *If R is a-symmetric, then so is $R_f^{(\Sigma,\mathbf{A})}$.*

However, *a*-transitivity is not in general preserved.

Transitive filtration

$$D_t^{\mathbf{A}}([w],[v]) = \bigwedge \{V(\varphi \vee \Diamond\varphi, v) \to V(\Diamond\varphi, w) \mid \Diamond\varphi \in \Sigma\}$$

$$B_t^{\mathbf{A}}([w],[v]) = \bigwedge \{V(\Box\varphi, w) \to V(\varphi \wedge \Box\varphi, v) \mid \Box\varphi \in \Sigma\}$$

Transitive filtration

$$D_t^{\mathbf{A}}([w],[v]) = \bigwedge \{V(\varphi \vee \Diamond\varphi, v) \to V(\Diamond\varphi, w) \mid \Diamond\varphi \in \Sigma\}$$

$$B_t^{\mathbf{A}}([w],[v]) = \bigwedge \{V(\Box\varphi, w) \to V(\varphi \wedge \Box\varphi, v) \mid \Box\varphi \in \Sigma\}$$

Proposition

Let Σ be a subformula closed set, $\mathfrak{M} = (W, D, B, V)$ an a-transitive $\mathbf{A}$-model, and $\mathfrak{M}_t^{(\Sigma,\mathbf{A})} = (W_\Sigma^{\mathbf{A}}, \mathrm{D}_t^{(\Sigma,\mathbf{A})}, \mathrm{B}_t^{(\Sigma,\mathbf{A})}, V_\Sigma^{\mathbf{A}})$, where $D_t^{(\Sigma,\mathbf{A})}$ and $B_t^{(\Sigma,\mathbf{A})}$ are obtained as described above. Then $\mathfrak{M}_t^{(\Sigma,\mathbf{A})}$ is a filtration of $\mathfrak{M}$ through Σ over $\mathbf{A}$. Moreover, $\mathfrak{M}_t^{(\Sigma,\mathbf{A})}$ is a-transitive.

Many-valued modal logics with the FMP

In what follows, we restrict our attention to **A**-frames (W, D, B) where $D = B = R$.

Many-valued modal logics with the FMP

Moreover, we will assume that **A** is a finite Heyting algebra.

Many-valued modal logics with the FMP

Lemma

The canonical models of $\mathbf{T}(\mathbf{A}), \mathbf{D}(\mathbf{A}), \mathbf{B}(\mathbf{A})$, $\mathbf{K4}(\mathbf{A})$ *are* 1*-reflexive, 1-serial, 1-symmetric and* 1*-transitive, respectively.*

Many-valued modal logics with the FMP

Lemma

The canonical models of $\mathbf{T}(\mathbf{A}), \mathbf{D}(\mathbf{A}), \mathbf{B}(\mathbf{A})$, $\mathbf{K4}(\mathbf{A})$ *are* 1*-reflexive, 1-serial, 1-symmetric and* 1*-transitive, respectively.*

Theorem

$\mathbf{T}(\mathbf{A}), \mathbf{D}(\mathbf{A}), \mathbf{B}(\mathbf{A})$ *and* $\mathbf{K4}(\mathbf{A})$ *are characterized by the classes of all finite* 1*-reflexive, 1-serial, 1-symmetric and* 1*-transitive* $\mathbf{A}$*-frames, respectively.* $\mathbf{S4}(\mathbf{A})$ *is characterized by the class of all finite* 1*-reflexive and* 1*-transitive* $\mathbf{A}$*-frames.*

A many-valued version of **GL**

Fix a Heyting chain $\mathbf{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ and consider the many valued modal language containing only $\Box$.

A many-valued version of **GL**

Transitivity axiom:

$$\Box p \to \Box\Box p \tag{4}$$

Löb formula:

$$\Box(\Box p \to p) \to \Box p \tag{gl}$$

A many-valued version of **GL**

Transitivity axiom:

$$\Box p \to \Box\Box p \tag{4}$$

Löb formula:

$$\Box(\Box p \to p) \to \Box p \tag{gl}$$

Let **GL**(**A**) be the logic obtained by adding **gl** and **4** as axioms to the axiomatization given by Bou et. al. (2011).

A many-valued version of **GL**

Transitivity axiom:

$$\Box p \to \Box\Box p \tag{4}$$

Löb formula:

$$\Box(\Box p \to p) \to \Box p \tag{gl}$$

Let **GL**(**A**) be the logic obtained by adding **gl** and **4** as axioms to the axiomatization given by Bou et. al. (2011).

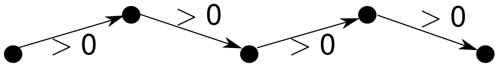
Possible interpretation: Provability in a lattice (or poset) of logics, e.g. axiomatic extensions of Peano Arithmetic. (Analogous to Fitting's multiple expert semantics.)

Frames for $\mathbf{GL}(\mathbf{A})$

A non-0 path in $\mathfrak{F} = (W, R)$ is a sequence $\langle w_0, w_1, \ldots \rangle$ such that $R(w_i, w_{i+1}) > 0$ for all $i \geq 0$.

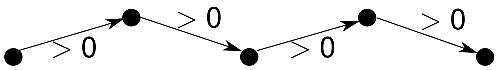
Frames for **GL(A)**

A non-0 path in $\mathfrak{F} = (W, R)$ is a sequence $\langle w_0, w_1, \ldots \rangle$ such that $R(w_i, w_{i+1}) > 0$ for all $i \geq 0$.



Frames for $\mathbf{GL}(\mathbf{A})$

A non-0 path in $\mathfrak{F} = (W, R)$ is a sequence $\langle w_0, w_1, \ldots \rangle$ such that $R(w_i, w_{i+1}) > 0$ for all $i \geq 0$.



Proposition

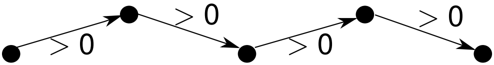
*$\mathfrak{F}, w \Vdash^1$ **gl** if, and only if:*

1. $\mathfrak{F}$ is 1-transitive, and

2. there are no infinite non-0 paths starting from w.

Frames for **GL**(**A**)

A non-0 path in $\mathfrak{F} = (W, R)$ is a sequence $\langle w_0, w_1, \ldots \rangle$ such that $R(w_i, w_{i+1}) > 0$ for all $i \geq 0$.



Proposition

*$\mathfrak{F}, w \Vdash^1$ **gl** if, and only if:*
1. $\mathfrak{F}$ is 1-transitive, and
2. there are no infinite non-0 paths starting from w.

Lemma

*Axiom **4** is canonical for 1-transitivity.*

Finite model property for $\mathbf{GL}(\mathbf{A})$

Theorem

$\mathbf{GL}(\mathbf{A})$ *is determined by the class of finite* 1*-transitive* $\mathbf{A}$*-frames, with no infinite non-*0 *paths.*

Finite model property for $\mathbf{GL}(\mathbf{A})$

Theorem

$\mathbf{GL}(\mathbf{A})$ *is determined by the class of finite* 1*-transitive* $\mathbf{A}$*-frames, with no infinite non-*0 *paths.*

Proof sketch:

Soundness by lemmas above.

Finite model property for $\mathbf{GL}(\mathbf{A})$

Theorem

$\mathbf{GL}(\mathbf{A})$ *is determined by the class of finite* 1*-transitive* $\mathbf{A}$*-frames, with no infinite non-*0 *paths.*

Proof sketch:

Soundness by lemmas above.

Suppose $\nvdash_{\mathbf{GL}(\mathbf{A})} \varphi$.

Finite model property for $\mathbf{GL}(\mathbf{A})$

Theorem

$\mathbf{GL}(\mathbf{A})$ *is determined by the class of finite* 1*-transitive* $\mathbf{A}$*-frames, with no infinite non-*0 *paths.*

Proof sketch:

Soundness by lemmas above.

Suppose $\nvdash_{\mathbf{GL}(\mathbf{A})} \varphi$.

Then $V^{\mathbf{GL}(\mathbf{A})}(\varphi, w) < 1$ for some w in the canonical model $\mathfrak{M}^{\mathbf{GL}(\mathbf{A})}$ (Bou et. al.).

Finite model property for **GL**(**A**)

Theorem

GL(**A**) *is determined by the class of finite* 1*-transitive* **A***-frames, with no infinite non-*0 *paths.*

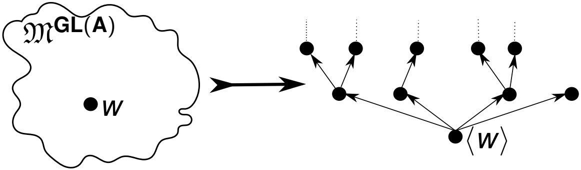
Proof sketch:

Soundness by lemmas above.

Suppose $\nvdash_{\mathbf{GL(A)}} \varphi$.

Then $V^{\mathbf{GL(A)}}(\varphi, w) < 1$ for some w in the canonical model $\mathfrak{M}^{\mathbf{GL(A)}}$ (Bou et. al.).

Take transitive unravelling of $\mathfrak{M}^{\mathbf{GL(A)}}$ around w:



Â

Finite model property for **GL**(**A**) *(cont.)*

We obtain a model $\mathfrak{M} = (W, R, V)$, where:

- W is the set of all finite non-0 paths in $\mathfrak{M}^{\mathbf{GL}(\mathbf{A})}$ starting at w;
- for all $\sigma_1, \sigma_2 \in W$, we have $R(\sigma_1, \sigma_2) = R^{\mathbf{GL}(\mathbf{A})}(\ell(\sigma_1, \ell(\sigma_2))$, where $\ell(\sigma)$ denotes the last element of σ, if $\sigma_2 = \langle \sigma_1, \sigma_3 \rangle$ for some non-0 path σ_3, otherwise $R(\sigma_1, \sigma_2) = 0$;
- $V(p, \sigma) = V^{\mathbf{GL}(\mathbf{A})}(p, \ell(\sigma))$ for all $p \in \Phi$ and $\sigma \in W$.

Finite model property for **GL(A)** *(cont.)*

We obtain a model $\mathfrak{M} = (W, R, V)$, where:

- W is the set of all finite non-0 paths in $\mathfrak{M}^{\mathbf{GL(A)}}$ starting at w;
- for all $\sigma_1, \sigma_2 \in W$, we have $R(\sigma_1, \sigma_2) = R^{\mathbf{GL(A)}}(\ell(\sigma_1, \ell(\sigma_2))$, where $\ell(\sigma)$ denotes the last element of σ, if $\sigma_2 = \langle \sigma_1, \sigma_3 \rangle$ for some non-0 path σ_3, otherwise $R(\sigma_1, \sigma_2) = 0$;
- $V(p, \sigma) = V^{\mathbf{GL(A)}}(p, \ell(\sigma))$ for all $p \in \Phi$ and $\sigma \in W$.

Finite model property for $\mathbf{GL}(\mathbf{A})$ *(cont.)*

We obtain a model $\mathfrak{M} = (W, R, V)$, where:

- W is the set of all finite non-0 paths in $\mathfrak{M}^{\mathbf{GL}(\mathbf{A})}$ starting at w;
- for all $\sigma_1, \sigma_2 \in W$, we have $R(\sigma_1, \sigma_2) = R^{\mathbf{GL}(\mathbf{A})}(\ell(\sigma_1, \ell(\sigma_2)),$ where $\ell(\sigma)$ denotes the last element of σ, if $\sigma_2 = \langle \sigma_1, \sigma_3 \rangle$ for some non-0 path σ_3, otherwise $R(\sigma_1, \sigma_2) = 0$;
- $V(p, \sigma) = V^{\mathbf{GL}(\mathbf{A})}(p, \ell(\sigma))$ for all $p \in \Phi$ and $\sigma \in W$.

Finite model property for $\mathbf{GL}(\mathbf{A})$ *(cont.)*

We obtain a model $\mathfrak{M} = (W, R, V)$, where:

- W is the set of all finite non-0 paths in $\mathfrak{M}^{\mathbf{GL}(\mathbf{A})}$ starting at w;
- for all $\sigma_1, \sigma_2 \in W$, we have $R(\sigma_1, \sigma_2) = R^{\mathbf{GL}(\mathbf{A})}(\ell(\sigma_1, \ell(\sigma_2))$, where $\ell(\sigma)$ denotes the last element of σ, if $\sigma_2 = \langle \sigma_1, \sigma_3 \rangle$ for some non-0 path σ_3, otherwise $R(\sigma_1, \sigma_2) = 0$;
- $V(p, \sigma) = V^{\mathbf{GL}(\mathbf{A})}(p, \ell(\sigma))$ for all $p \in \Phi$ and $\sigma \in W$.

Finite model property for **GL(A)** *(cont.)*

We obtain a model $\mathfrak{M} = (W, R, V)$, where:

- W is the set of all finite non-0 paths in $\mathfrak{M}^{\mathbf{GL(A)}}$ starting at w;
- for all $\sigma_1, \sigma_2 \in W$, we have $R(\sigma_1, \sigma_2) = R^{\mathbf{GL(A)}}(\ell(\sigma_1, \ell(\sigma_2))$, where $\ell(\sigma)$ denotes the last element of σ, if $\sigma_2 = \langle \sigma_1, \sigma_3 \rangle$ for some non-0 path σ_3, otherwise $R(\sigma_1, \sigma_2) = 0$;
- $V(p, \sigma) = V^{\mathbf{GL(A)}}(p, \ell(\sigma))$ for all $p \in \Phi$ and $\sigma \in W$.

Lemma

For all formulas ψ and all $\sigma \in W$, we have $V(\psi, \sigma) = V^{\mathbf{GL(A)}}(\psi, \ell(\sigma))$. Moreover, $\mathfrak{M}$ is 1-transitive.

Finite model property for **GL**(**A**) *(cont.)*

We obtain a model $\mathfrak{M} = (W, R, V)$, where:

- W is the set of all finite non-0 paths in $\mathfrak{M}^{\mathbf{GL}(\mathbf{A})}$ starting at w;
- for all $\sigma_1, \sigma_2 \in W$, we have $R(\sigma_1, \sigma_2) = R^{\mathbf{GL}(\mathbf{A})}(\ell(\sigma_1, \ell(\sigma_2))$, where $\ell(\sigma)$ denotes the last element of σ, if $\sigma_2 = \langle \sigma_1, \sigma_3 \rangle$ for some non-0 path σ_3, otherwise $R(\sigma_1, \sigma_2) = 0$;
- $V(p, \sigma) = V^{\mathbf{GL}(\mathbf{A})}(p, \ell(\sigma))$ for all $p \in \Phi$ and $\sigma \in W$.

Lemma

For all formulas ψ and all $\sigma \in W$, we have $V(\psi, \sigma) = V^{\mathbf{GL}(\mathbf{A})}(\psi, \ell(\sigma))$. Moreover, $\mathfrak{M}$ is 1-transitive.

$$\therefore V(\varphi, \langle w \rangle) = V^{\mathbf{GL}(\mathbf{A})}(\varphi, w) < 1 \text{ and } \mathfrak{M} \Vdash^1 \mathbf{gl}$$

Finite model property for $\mathbf{GL}(\mathbf{A})$ *(cont.)*

Construct a sequence of models $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ such that for all $i \geq 1$, $\mathfrak{M}_i$ is a submodel of $\mathfrak{M}$.

Finite model property for **GL**(**A**) *(cont.)*

Construct a sequence of models $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ such that for all $i \geq 1$, $\mathfrak{M}_i$ is a submodel of $\mathfrak{M}$.
Set $W_1 = \{\langle w \rangle\}$.

Finite model property for $\mathbf{GL}(\mathbf{A})$ *(cont.)*

Construct a sequence of models $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ such that for all $i \geq 1$, $\mathfrak{M}_i$ is a submodel of $\mathfrak{M}$.
Set $W_1 = \{\langle w \rangle\}$.
Defect of $v \in W_i$: formula $\Box\psi \in \mathit{SubFml}(\varphi)$ such that $V(\Box\psi, v) = \beta < 1$ and there is no $u \in W_i$ such that $R(v, u) \rightarrow V(\psi, u) = \beta$.

Finite model property for $\mathbf{GL}(\mathbf{A})$ *(cont.)*

Construct a sequence of models $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ such that for all $i \geq 1$, $\mathfrak{M}_i$ is a submodel of $\mathfrak{M}$.
Set $W_1 = \{\langle w \rangle\}$.
Defect of $v \in W_i$: formula $\Box\psi \in \mathit{SubFml}(\varphi)$ such that $V(\Box\psi, v) = \beta < 1$ and there is no $u \in W_i$ such that $R(v, u) \to V(\psi, u) = \beta$.
Repair defects using:

Lemma
*Let $\mathfrak{N} = (X, S, U)$ be an $\mathfrak{A}$-model, and suppose $\mathfrak{N} \Vdash^1$ **gl** and $v \in X$. If $U(\Box\psi, v) = \beta < 1$, then there is a state $u \in X$ such that $S(v, u) \to U(\psi, u) = \beta$ and $U(\Box\psi, u) > \beta$, and moreover, $u \neq v$.*

Finite model property for **GL(A)** *(cont.)*

Construct a sequence of models $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$ such that for all $i \geq 1$, $\mathfrak{M}_i$ is a submodel of $\mathfrak{M}$.

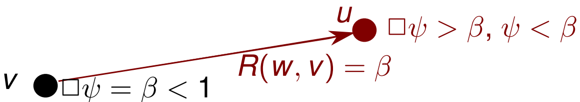
Set $W_1 = \{\langle w \rangle\}$.

Defect of $v \in W_i$: formula $\Box\psi \in SubFml(\varphi)$ such that $V(\Box\psi, v) = \beta < 1$ and there is no $u \in W_i$ such that $R(v, u) \to V(\psi, u) = \beta$.

Repair defects using:

Lemma

*Let $\mathfrak{N} = (X, S, U)$ be an $\mathfrak{A}$-model, and suppose $\mathfrak{N} \Vdash^1$ **gl** and $v \in X$. If $U(\Box\psi, v) = \beta < 1$, then there is a state $u \in X$ such that $S(v, u) \to U(\psi, u) = \beta$ and $U(\Box\psi, u) > \beta$, and moreover, $u \neq v$.*



Finite model property for $\mathbf{GL}(\mathbf{A})$ *(cont.)*

Claim

The construction terminates, having produced a finite model $\mathfrak{M}_n$ in which no point has a defect.

Finite model property for $\mathbf{GL}(\mathbf{A})$ *(cont.)*

Claim

The construction terminates, having produced a finite model $\mathfrak{M}_n$ in which no point has a defect.

Claim

$V_n(\psi, v) = V(\psi, v)$, for each formula $\psi \in SubFml(\varphi)$ and point $v \in W_n$.

References

[Bou et. al.] *On the minimum many-valued modal logic over a finite residuated lattice*, *Journal of Logic and Computation*, 21:739–790, (2011).
[M. Fitting] *How True It Is = Who Says It's True*, *Studia Logica*, 91(3):335-366, (2009).

Thank you!