Unification in first-order logics: superintuitionistic (and modal)

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$\mathbf{0}$: some extensions of KC (Ghilardi), modal l. K (Jeřabek).

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EXAM. Classical PC: $\varepsilon_A(p) = (\neg A \vee p) \wedge (A \vee \tau(p))$, τ is a ground unifier for A, so called Löwenheim substitution (reproductive solut.)

Discriminator var., Modal S5, NExt S4.3 (DW), unitar not proj KC

Applications: Admissible rules, (A)SC

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EX. the Harrop rule $\dfrac{\neg A \to B_1 \vee B_2}{(\neg A \to B_1) \vee (\neg A \to B_2)}$ is admissible in INT

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We consider a first-order (or predicate) *intuitionistic* language without function letters.

free individual variables: $a_1, a_2, a_3, \ldots$

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q-Fm denotes the set of all quasi-formulas, (*Fm* - formulas).

$\varphi \in Fm$ iff $\varphi \in q\text{-}Fm$ and bound variables in φ do not occur free.

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$$\varepsilon(A \to B) = \varepsilon(A) \to \varepsilon(B); \qquad \varepsilon(A \wedge B) = \varepsilon(A) \wedge \varepsilon(B);$$

$$\varepsilon(\neg A) = \neg\varepsilon(A); \qquad \varepsilon(A \vee B) = \varepsilon(A) \vee \varepsilon(B);$$

$$\varepsilon(\forall_x A) = \forall_x \varepsilon(A) \qquad \varepsilon(\exists_x A) = \exists_x \varepsilon(A)$$

$\varepsilon(P_j(x_1,\ldots,x_k)) \neq P_j(x_1,\ldots,x_k)$ for a finite number of P_j's.

$=$ is defined here up to a correct renaming of bound variables in the substituted formulas: operation $(A)_n$ - renamig bound var. in a uniform way.

• Pogorzelski, W.A., Prucnal, T., *Structural completeness of the first-order predicate calculus*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 21 (1975), 315-320.

$fv(\varepsilon(A)) \subseteq fv(A)$ we remove this condition !!

•Church, A., *Introduction to Mathematical Logic I*, Princeton 1956

Pogorzelski, Prucnal: Classical Predicate Logic is not SC;

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Example: Non-unifiable but Consistent (1 predicate variable P):
$\exists_x \neg P(x) \land \exists_x P(x)$, $\exists_x \neg P(x) \land \neg\neg\exists_x P(x)$, $\neg\forall_x P(x) \land \neg\neg\exists_x P(x)$,

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The rule A/B is called *passive* in L, if A is not unifiable in L.
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which means that all passive rules are derivable in the extension of Q–INT *with the rules* $P\forall$*.*

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Structural completeness, SC, is too strong for predicate logics. It should be replaced by *Almost SC,* ASC , which is more suitable

Projective formulas and Harrop formulas

A formula A is *L-projective* in a superintuitionistic predicate logic L if there is a substitution ε (for predicate variables) called a *projective unifier* for A, such that $\vdash_L \varepsilon(A)$ and

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Harrop q-formulas $q\text{-}Fm_H$ (*Harrop formulas* Fm_H) are defined by:

1. all elementary q-formulas (including $\bot$) are Harrop q-formulas;
2. if $A, B \in q\text{-}Fm_H$, then $A \wedge B \in q\text{-}Fm_H$;
3. if $B \in q\text{-}Fm_H$, then $A \to B \in q\text{-}Fm_H$;
4. if $B \in q\text{-}Fm_H$, then $\forall_{x_j} B \in q\text{-}Fm_H$.

Neither disjunction nor existential q-formula is a Harrop formula.

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Disjunction and Existential Property

Let L be a predicate logic and A be L-projective.

Theorem

(i) *if* $\vdash_L A \to B_1 \vee B_2$, *then* $\vdash_L (A \to B_1) \vee (A \to B_2)$;
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L has the *disjunction property (DP)* if $\vdash_L B_1 \vee B_2$ implies either $\vdash_L B_1$, or $\vdash_L B_2$. The logic has the *existence property (EP)* if $\vdash_L \exists_x C(x)$ implies $\vdash_L C(t)$ for some term (=free variable) t.

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The rule

$$\frac{\neg A \to B_1 \vee B_2}{(\neg A \to B_1) \vee (\neg A \to B_2)}$$

Rules admissible in superintuitionistic logics

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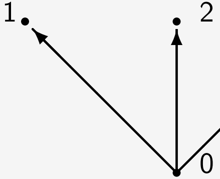

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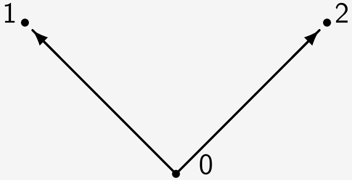

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Logics with Projective Unification

Theorem

The following conditions are equivalent

(i) L enjoys projective unification;

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Definitions (valid in *P.Q–LC*)

$$A \vee B := ((A \to B) \to B) \wedge ((B \to A) \to A);$$

$$\exists_x A(x) := \forall_x(\forall_y(A(y) \to A(x)) \to A(x)).$$

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Moreover, if we extend the $\{\to, \wedge, \bot, \forall\}$ fragment of *Q-INT* with the above definitions, we obtain *P.Q–LC*.

Corollary

P.Q–LC is the least predicate logic in which $A \vee B$ and $\exists_x A(x)$ are defined in $\{\to, \wedge, \bot, \forall\}$ (or are Harrop's).

Corollary

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P *is valid on a rooted frame* $\mathfrak{F} =< W, \leq, \mathcal{D} >$ *if and only if* $\mathfrak{F}$ *has a constant domain and one of the following holds*

(1) the domain of $\mathfrak{F}$ *is one-element;*

(2) the domain of $\mathfrak{F}$ *is finite and* $\leq$ *is a linear order on* W*;*

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The logic P.Q–LC is Kripke incomplete.

Unification types.

We develop unification types for superintutionistic predicate logics. Standard definitions of the types: $1, \omega, \infty, 0$ are introduced but if one tries to follow the results on unification types in propositional logics, despite some similarities, the results are different: the unification type of Q–L is usually "more complicated" then the unification type of the propositional logic L.

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Unification in L is *unitary* (type is 1) if the set of unifiers of any unifiable formula A contains a greatest, w.r.t $\preceq$, element of A, an mgu of A). Unification in L is finitary (type is ω), if it is not 1 but there is finitely many $\preceq$-maximal unifiers for each unifiable A and each unifier for A is bounded by a maximal one. Unification in L is infinitary (type is ∞) if it is not 1, nor ω, and each L-unifier of A is bounded by a maximal one. Unification in L is nulary (type is 0) if it is neither 1, nor ω, nor ∞.

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Corollary

Unification in P.Q–LC and all its extensions is unitary.

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For every superintuitionistic predicate logic L
(i) if Q–KC $\subseteq$ L, then unification in L is unitary or nullary;
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Unification in Q–LC as well as in Q–KC, is nullary (in propos. $\mathbf{1}$*)*

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Conjecture: some predicate logics have infinitary unification.

Thank you for your attention !

Theorem

The rules $\mathrm{P}_{\Diamond\exists}$:

$$\frac{\Diamond A \wedge \Diamond\neg A}{\bot}, \frac{\Diamond\exists_z A(z) \wedge \Diamond\exists_z\neg A(z)}{\bot}, \frac{\Diamond\exists_u\exists_v A(u,v) \wedge \Diamond\exists_u\exists_v\neg A(u,v)}{\bot}, \ldots$$

form a basis for all passive rules over Q–S4 *and its extensions. No sublogic of* Q-CL *is structurally complete (too strong property).*

Projective fromulas in Predicate Modal Logics

A unifier ε for predicate variables is *projective* for a formula A (or formula A is *projective*) in a logic L if

$$\vdash_L \Box A \to \forall_{x_1} \cdots \forall_{x_k} \big(\varepsilon(P_j(x_1, \ldots, x_k)) \leftrightarrow P_j(x_1, \ldots, x_k)\big), \text{ for each } P_j.$$

Theorem

For any L*-projective formula A and any formulas $B_1, B_2, \exists_x C(x)$,*
(i) if $\vdash_L A \to \Box B_1 \vee \Box B_2$, *then* $\vdash_L \Box(\Box A \to B_1) \vee \Box(\Box A \to B_2)$;
(ii) if $\vdash_L A \to \exists_x \Box C(x)$, *then* $\vdash_L \exists_x \Box(\Box A \to C(x))$.

the *disjunction property (DP)*: $\vdash_L \Box B_1 \vee \Box B_2 \Rightarrow \vdash_L B_1$, or $\vdash_L B_2$.
the *existence property (EP)*: $\vdash_L \exists_x \Box C(x) \Rightarrow \vdash_L C(t)$, for some term t (free variable)
Rasiowa-Sikorski: Q–S4 enjoys *(DP)* and *(EP)*;

Corollary

For L *with (DP) and (EP), any* L*-projective A, any $B_1, B_2, \exists_x C(x)$:*
(i) if $\vdash_L A \to \Box B_1 \vee \Box B_2$, then $\vdash_L (\Box A \to B_1)$ or $\vdash_L (\Box A \to B_2)$;
(ii) if $\vdash_L A \to \exists_x \Box C(x)$, then $\vdash_L \Box A \to C(t)$ for some t.

Formulas which are not projective (in Q–S4):
- any $\Box B_1 \vee \Box B_2$ which does not reduce to any its disjunct, or
- any $\exists_x \Box C(x)$ which does not collapse to any its instance $\Box C(t)$.

Projective unification, ASC in Predicate Modal Logics

Corollary

If L *enjoys projective unification, then* P.Q–S4.3$\subseteq$ L, *where* P *:* $\exists_x \Box(\exists_x \Box P(x) \to P(x))$.
(The converse - if $=$ *is in the language).*

Q–S5 has projective unification.
P.Q–S4.3 is Kripke incomplete. BF :$\forall_x \Box A \to \Box \forall_x A \notin$ P.Q–S4.3;
- P $\notin$ BF.Q–S4.3 and .3 $\notin$ P.Q–S.4.

Corollary

If L *has projective unification, then* L *is Almost Structurally Complete (ASC).*
Q–S5 *is ASC.*

Let $\Box^+ A = A \wedge \Box A$ and $\Diamond^+ A = A \vee \Diamond A$.
Ghilardi and Sacchetti (JSL68,2004): For L a prop. modal logic L $\subseteq$ K4, unification in L is filtering iff : $2^+ : \quad \Diamond^+\Box^+ A \to \Box^+\Diamond^+ A$.

Theorem

Let L *be a predicate modal logics extending* Q–K4*. Unification in* L *is filtering iff* L *contains* $\quad 2^+ : \quad \Diamond^+\Box^+ A \to \Box^+\Diamond^+ A$*.*

Corollary

(i) For every predicate modal logic L constaining Q–K4
if $2^+ : \quad \Diamond^+\Box^+ A \to \Box^+\Diamond^+ A$ *is in* L *, then unification in* L *is unitary or nullary. Moreover, if* L *enjoys unitary unification, then* $\Diamond^+\Box^+ A \to \Box^+\Diamond^+ A$ *is in* L*, i.e.* Q–K4.$2^+ \subseteq$ L*.*
(ii) For every predicate modal logic L containing Q–S4
if $2 : \quad \Diamond\Box A \to \Box\Diamond A$ *is in* L*, then unification in* L *is unitary or nullary. Moreover, if* L *enjoys unitary unification, then* $\Diamond\Box A \to \Box\Diamond A$ *is in* L*, i.e.* Q–S4.2$\subseteq$ L*.*

Unification types in Predicate Modal Logics

a predicate modal logic L constaining Q–S4 have the *weak existence property*, *(WEP)*, if $\exists_x \Box A(x) \in \mathsf{L}$ implies $\Box A(t_1) \vee \cdots \vee \Box A(t_n) \in \mathsf{L}$, for some terms $t_1, \ldots, t_n$.

Theorem

If a modal predicate logic L *enjoys (WEP), then unification in* L *is neither finitary, nor unitary.*

Corollary

Unification in Q–S4.3 *and in* Q–S4.2 *is nullary.*

In contrast to S4.3 unification in some extensions of Q–S4.3 can be unitary or nullary.

Corollary

The unification type of Q–K4 *and* Q–S4 *is either* 0 *or* ∞*.*