Structure theorem for a class of group-like residuated chains à la Hahn

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FL-algebras

An algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, \mathsf{t}, \mathsf{f})$ is called a *full Lambek algebra* or an *FL-algebra*, if

- $(A, \wedge, \vee)$ is a lattice (i.e., $\wedge$, $\vee$ are commutative, associative and mutually absorptive),
- $(A, \cdot, \mathsf{t})$ is a monoid (i.e., $\cdot$ is associative, with unit element t),
- $x \cdot y \leq z$ iff $y \leq x\backslash z$ iff $x \leq z/y$, for all $x, y, z \in A$,
- f is an arbitrary element of A.

Residuated lattices are exactly the f-free reducts of FL-algebras. So, for an FL-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, \mathsf{t}, \mathsf{f})$, the algebra $\mathbf{A}_r = (A, \wedge, \vee, \cdot, \backslash, /, \mathsf{t})$ is a residuated lattice and f is an arbitrary element of A. The maps $\backslash$ and $/$ are called the *left* and *right division*.

- commutative: $x \to y$

Group-like FL_e-algebras

- An FL_e-algebra is a commutative FL-algebra.
- An FL_e-**chain** is a totally ordered FL_e-algebra.
- An FL_e-algebra is called **involutive** if $x'' = x$ where $x' = x \to f$ (note that $f' = t$)
- An FL_e-algebra is called **group-like** if it is involutive and $f = t$

Hahn's Embedding Theorem

PARTIALLY ORDERED ALGEBRAIC SYSTEMS

L. FUCHS

Professor of Mathematics
L. Eötvös University
Budapest

1963

5. Hahn's embedding theorem

This section is devoted to the deepest result in the theory of f. o. Abelian groups. This asserts the embeddability of f. o. Abelian groups in the lexicographic product of real groups.

Theorem 16. (HAHN's Embedding Theorem, HAHN [1].) *Every f. o. vector space G over the rational number field is o-isomorphic to a subspace of the lexicographically ordered function space*[20] $W(G)$.

HAHN, H. [1] Über die nichtarchimedischen Grössensysteme, *S.-B. Akad. Wiss. Wien. IIa*, **116** (1907), 601—655.

The original proof of HAHN was extremely long and complicated. Recently, several authors have obtained simpler proofs and generalizations. The proof above is based on an idea of HAUSNER—WENDEL [1]: they proved HAHN's theorem for vector spaces over the real field and CLIFFORD [4] observed that their method works in the general case as well. For other proofs see BANASCHEWSKI [1], GRAVETT [2], RIBENBOIM [2], CONRAD [1], [7]. The last author has extended the theorem to certain p. o. Abelian groups and to even more general systems; he uses decompositions of the given group.

Recently, P. CONRAD, J. HARVEY and CH. HOLLAND proved HAHN's embedding theorem for commutative l. o. groups.

HAUSNER, M.—WENDEL, J. G. [1] Ordered vector spaces, *Proc. Amer. Math. Soc.*, **3** (1952), 977—982.

CLIFFORD, A. H.
— [4] Note on Hahn's theorem on ordered Abelian groups, *Proc. Amer. Math. Soc.*, **5** (1954), 860—863.

BANASCHEWSKI, B. [1] Totalgeordnete Moduln, *Archiv Math.*, **7** (1956), 430—440. — [2] Über die Vervollständigung geordneter Gruppen, *Math. Nachrichten*, **16** (1957), 51—71.

GRAVETT, K. A. H. — [2] Ordered Abelian groups, *Quart. Journ. Math. Oxford*, **7** (1956), 57—63.

RIBENBOIM, P. [2] Sur les groupes totalement ordonnés et l'arithmétique des anneaux de valuation, *Summa Brasil. Math.*, **4** (1958), 1—64. —[3] Sur quelques

CONRAD, P. [1] Embedding theorems for Abelian groups with valuations, *Amer. Journ. Math.*, **75** (1953), 1—29.
— [7] A note on valued linear spaces, *Proc. Amer. Math. Soc.*, **9** (1958), 646—647. — [8]

Comparison

- Hahn's theorem:
- Every totally ordered Abelian group embeds in a lexicographic product of real groups.

- Our embedding theorem:
- Every densely-ordered group-like FL_e-chain, which has finitely many idempotents embeds in a finite partial-lexicographic product of totally ordered Abelian groups.

A Few Other Related Results

Ordinal Sums

- Every naturally totally ordered, commutative semigroup is uniquely expressible as the ordinal sum of a totally ordered set of ordinally irreducible such semigroups

  [A. H. Clifford, Naturally totally ordered commutative semigroups, *Amer. J. Math.*, 76 vol. 3 (1954), 631–646.]

The Theory of Compact Semigroups

- Topological semigroups over compact manifolds with connected, regular boundary *B* such that *B* is a subsemigroup: a subclass of compact connected Lie groups and via classifying (I)-semigroups, that is, semigroups on arcs such that one endpoint functions as an identity for the semigroup, and the other functions as a zero.

  [P.S. Mostert, A.L. Shields, On the structure of semigroups on a compact manifold with boundary, *Ann. Math.*, 65 (1957), 117–143.]

The Theory of Compact Semigroups

- (I)-semigroups are ordinal sums of three basic multiplications which an arc may possess.

  The word ‘topological’ refers to the continuity of the semigroup operation with respect to the topology.

  [P.S. Mostert, A.L. Shields, On the structure of semigroups on a compact manifold with boundary, *Ann. Math.*, 65 (1957), 117–143.]

Structure of GBL-algebras

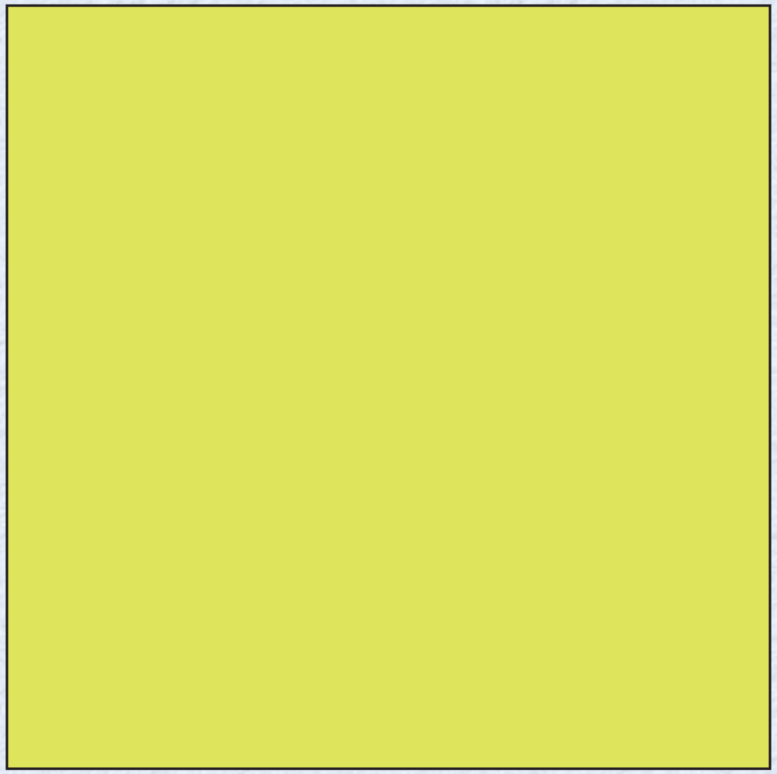
- BL-algebra = naturally ordered + semilinear integral residuated lattice
- BL-algebras are subdirect poset products of MV-chains and product chains.

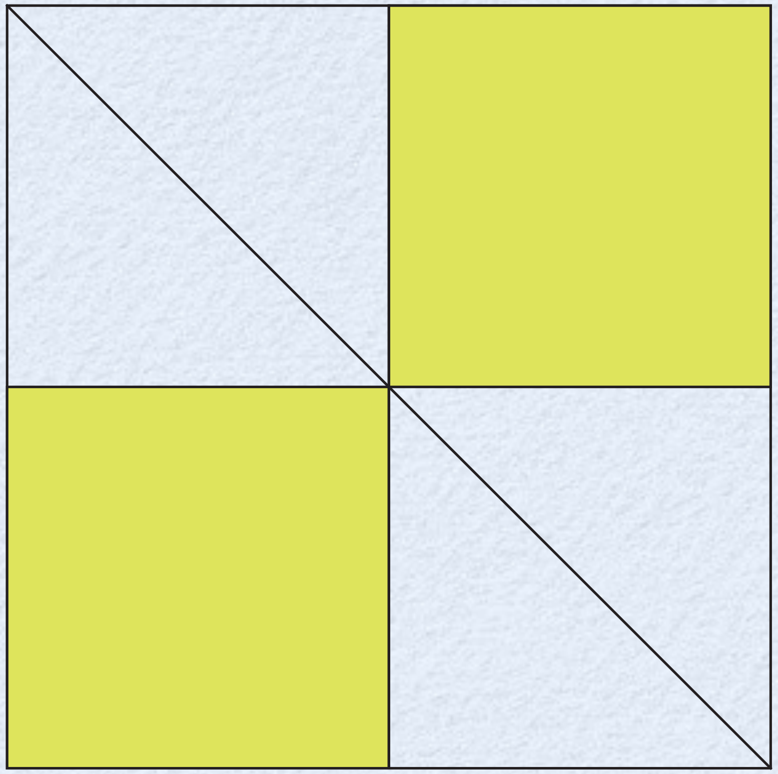
  [P Jipsen, F. Montagna, Embedding theorems for normal GBL-algebras, *Journal of Pure and Applied Algebra*, 214 (2010), 1559–1575.]

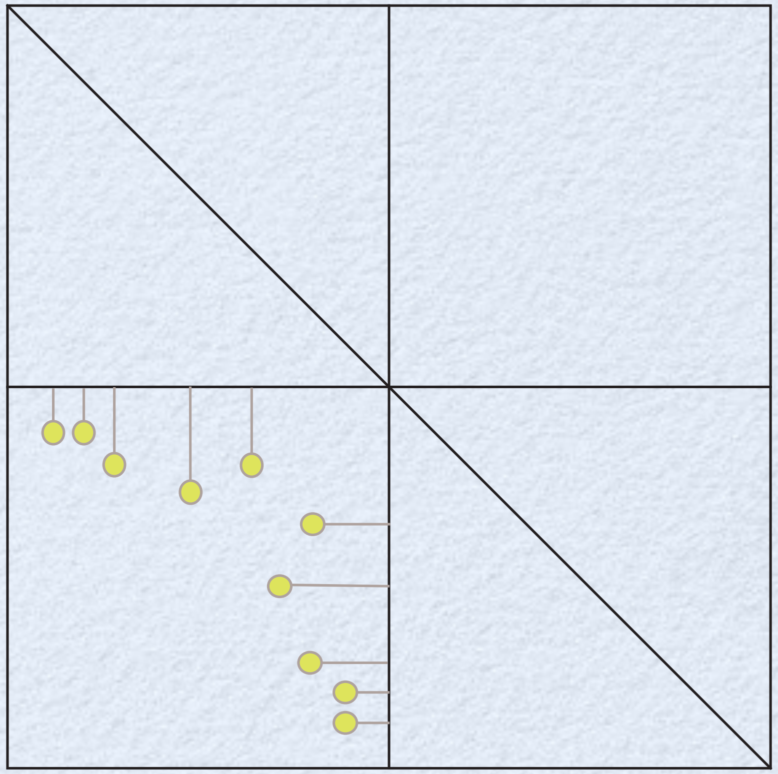
  (A generalization of the Conrad-Harvey-Holland representation)

Weakening the Naturally Ordered Property Entering the Non-integral Case

- [P Jipsen, F. Montagna, Embedding theorems for normal GBL-algebras, ***Journal of Pure and Applied Algebra***, Vol. 214. 1559–1575. (2010)]
- [SJ, F. Montagna, Strongly Involutive Uninorm Algebras ***Journal of Logic and Computation*** Vol. 23 (3), 707-726. (2013)]
- [SJ, F. Montagna, A classification of certain group-like FL_e -chains, ***Synthese*** Vol. 192 (7), 2095-2121. (2015)]

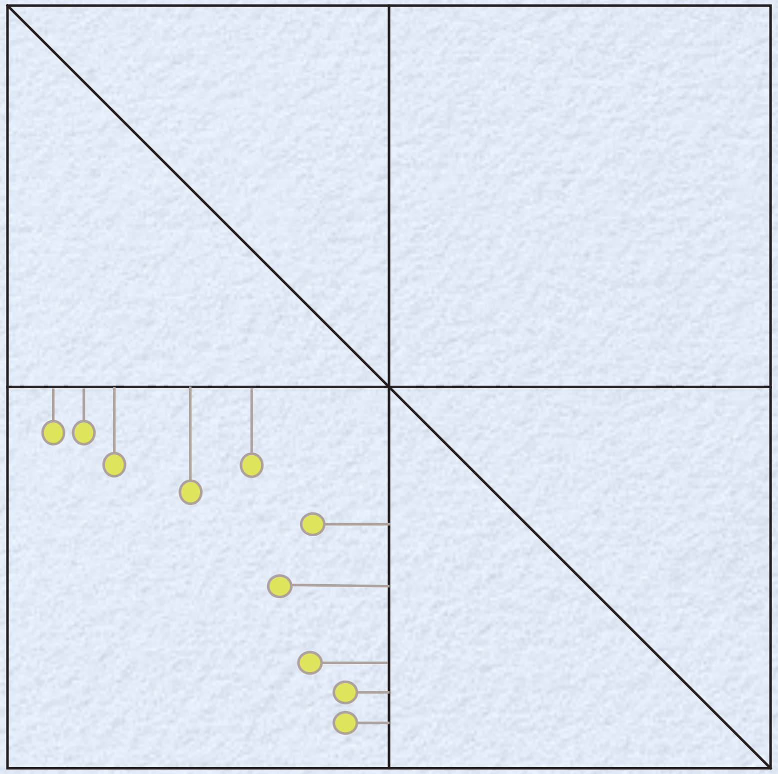






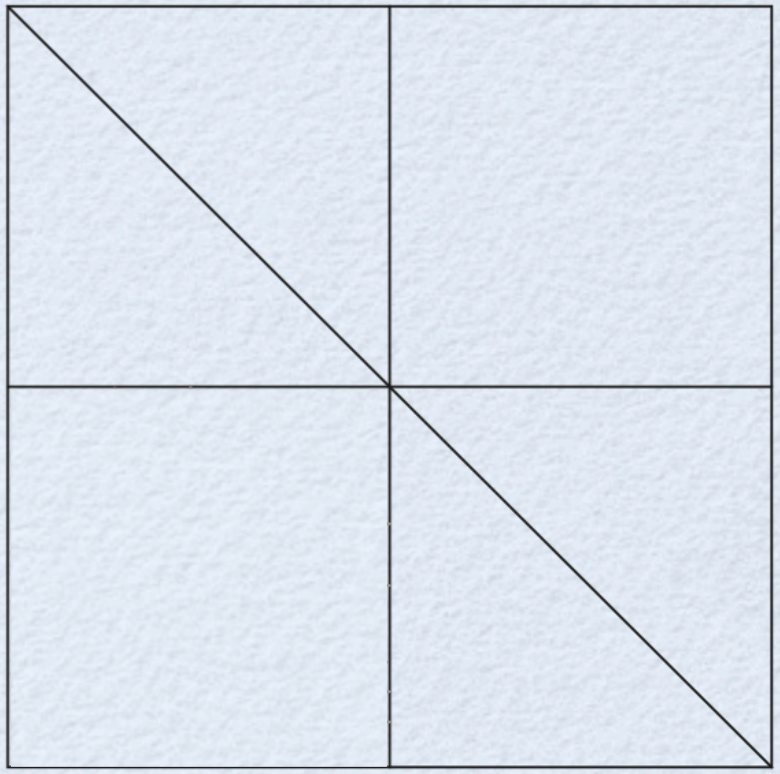
Absorbent Continuous Group-like Commutative Residuated Monoids on Complete and Order-dense Chains

- [SJ, F. Montagna, A classification of certain group-like FL_e -chains, ***Synthese*** Vol. 192 (7), 2095-2121. (2015)]

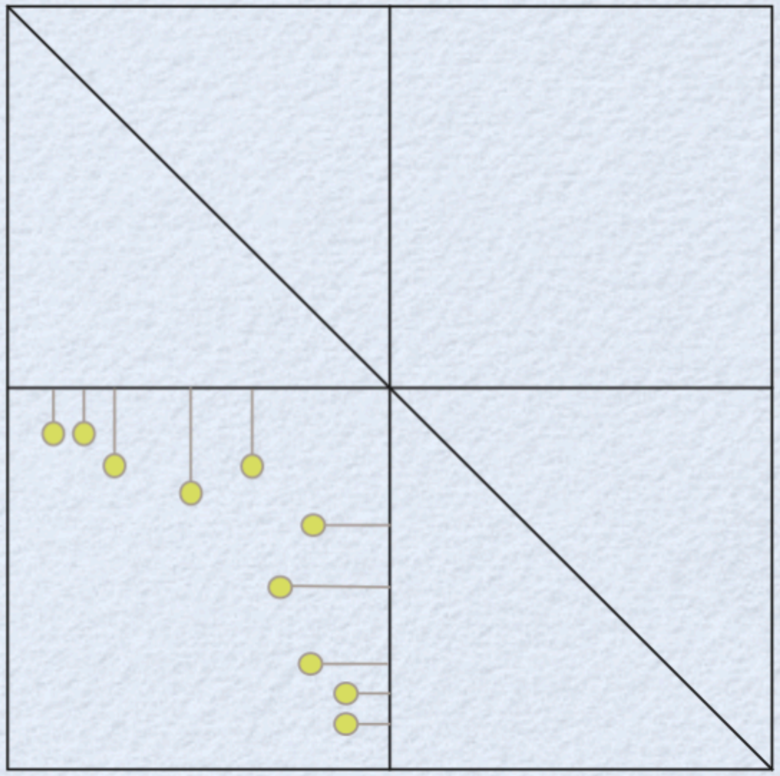


Absorbent Continuous Group-like Commutative Residuated Monoids on Complete and Order-dense Chains

- [SJ, Group Representation and Hahn-type Embedding for a Class of Residuated Monoids, (submitted)



- [SJ, F. Montagna, A classification of certain group-like FL_e -chains, ***Synthese*** Vol. 192 (7), 2095-2121. (2015)]



Group-like Commutative Residuated Monoids on Order-dense Chains

- [SJ, Group Representation and Hahn-type Embedding for a Class of Residuated Monoids, (submitted)

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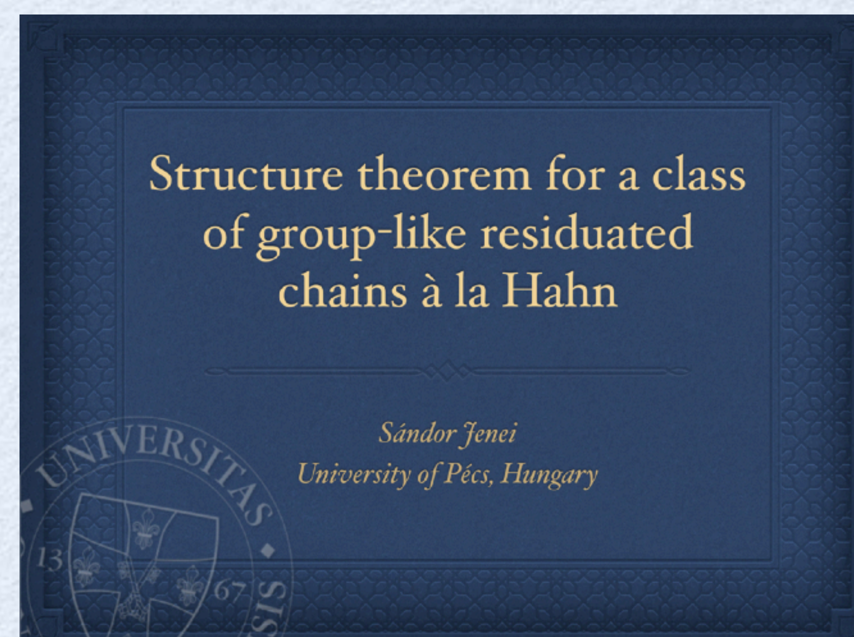

Group-like Commutative Residuated Monoids on Order-dense Chains

with finitely many idempotents

- [SJ, Group Representation and Hahn-type Embedding for a Class of Residuated Monoids, (submitted)

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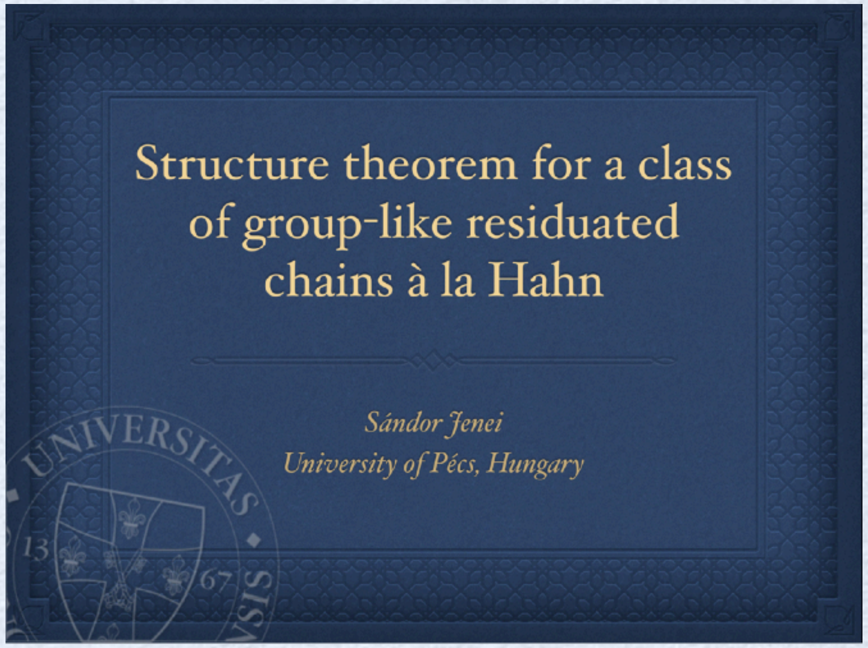


About the adjective "group-like" (t=f)

1. Conic representation of group-like FLe-algebras

- Conic representation: For any conic, IRL

$$x \circledast y = \begin{cases} x \oplus y & \text{if } x, y \in X^{+} \\ x \otimes y & \text{if } x, y \in X^{-} \\ (x \rightarrow_{\oplus} y')' & \text{if } x \in X^{+}, y \in X^{-}, \text{ and } x \leq y' \\ (y \rightarrow_{\otimes} x')' & \text{if } x \in X^{+}, y \in X^{-}, \text{ and } x \not\leq y' \\ (y \rightarrow_{\oplus} x')' & \text{if } x \in X^{-}, y \in X^{+}, \text{ and } x \leq y' \\ (x \rightarrow_{\otimes} y')' & \text{if } x \in X^{-}, y \in X^{+}, \text{ and } x \not\leq y' \end{cases}$$

- [S. Jenei, Structural description of a class of involutive uninorms via skew symmetrization, *Journal of Logic and Computation*, 21 vol. 5, 729–737 (2011)

2. Group-like FLe-algebras vs. lattice-ordered groups

SJ.

Theorem 2.5. *For a group-like FL_e-algebra $(X, \wedge, \vee, \circledast, \rightarrow_{\circledast}, t, f)$ the following statements are equivalent:*

(1) *Each element of X has inverse given by $x^{-1} = x'$, and hence $(X, \wedge, \vee, \circledast, t)$ is a lattice-ordered Abelian group,*

(2) *$\circledast$ is cancellative,*

(3) *$\tau(x) = t$ for all $x \in X$.* $\tau(x) = x \rightarrow x$

(4) *The only idempotent element in the positive cone of X is t.*

3. Representation of group-like FL_e-chains by groups and Hahn-type embedding

- Coming soon…

Partial-Lexicographic Products

Definition 1. *(Partial-lexicographic products)*
Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \rightarrow_*, t_X, f_X)$ be a group-like FL_e-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \star, \rightarrow_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement ${}'^*$ and ${}'^\star$, respectively.

Add a top element $\top$ to Y, and extend $\star$ by $\top \star y = y \star \top = \top$ for $y \in Y \cup \{\top\}$, then add a bottom element $\bot$ to $Y \cup \{\top\}$, and extend $\star$ by $\bot \star y = y \star \bot = \bot$ for $y \in Y \cup \{\bot, \top\}$.

Let $\mathbf{X}_1 = (X_1, \wedge_X, \vee_X, *, \rightarrow_*, t_X, f_X)$ be any cancellative subalgebra of $\mathbf{X}$ (by Theorem 1, $\mathbf{X}_1$ is a lattice ordered group). We define

$$\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y}^{\bot\top})} = \left(X_{\Gamma(X_1, Y^{\bot\top})}, \leq, \circledast, \rightarrow_\circledast, (t_X, t_Y), (f_X, f_Y) \right),$$

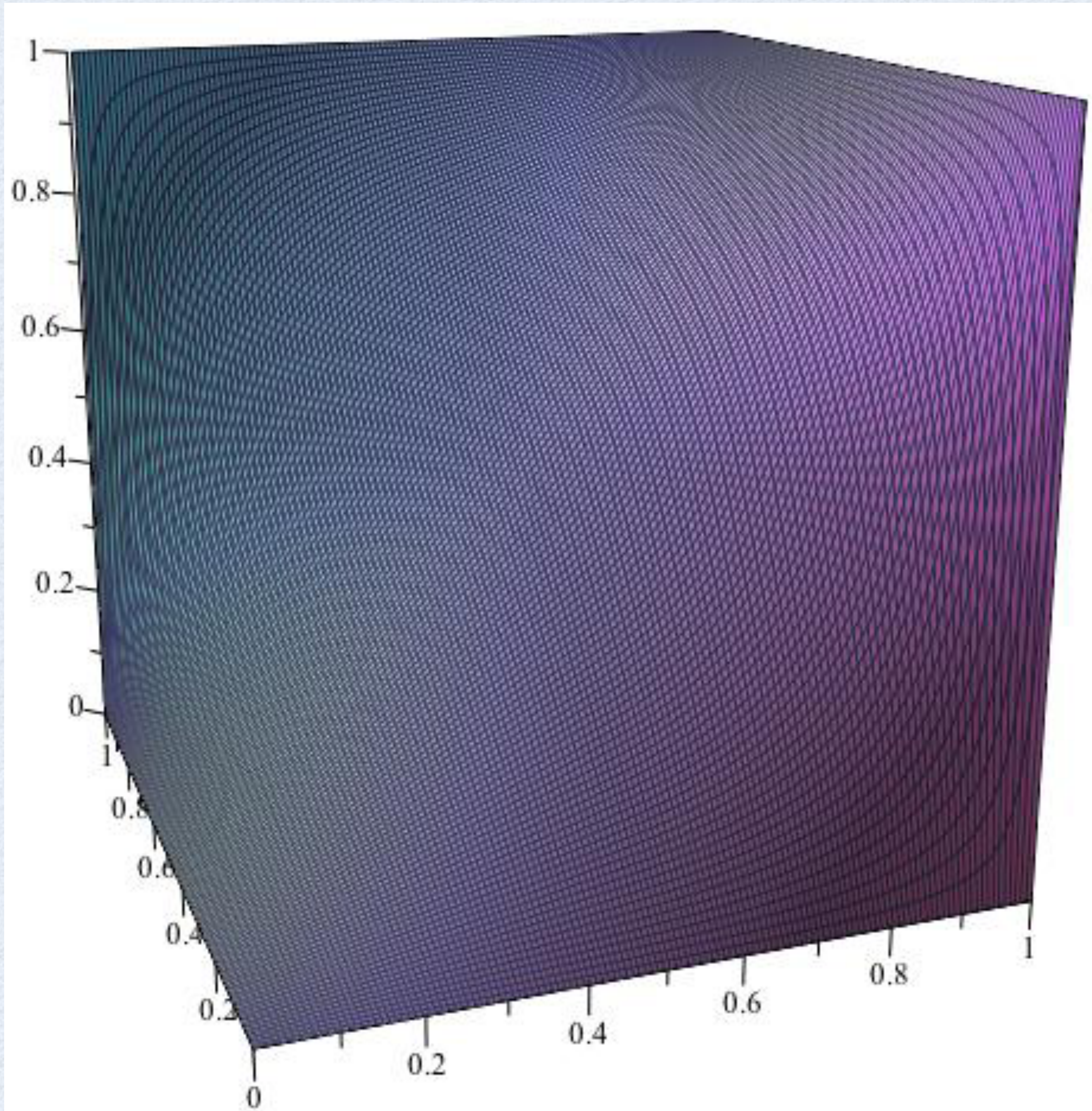
where

$$X_{\Gamma(X_1, Y^{\bot\top})} = (X_1 \times (Y \cup \{\bot, \top\})) \cup ((X \setminus X_1) \times \{\bot\}),$$

$\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y \cup \{\bot, \top\}}$ to $X_{\Gamma(X_1, Y^{\bot\top})}$, $\circledast$ is defined coordinatewise, and the operation $\rightarrow_\circledast$ is given by $(x_1, y_1) \rightarrow_\circledast (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ where

$$(x, y)' = \begin{cases} (x'^*, y'^\star) & \text{if } x \in X_1 \\ (x'^*, \bot) & \text{if } x \notin X_1 \end{cases}.$$

Call $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1}, \mathbf{Y}^{\bot\top})}$ the *(type-I) partial-lexicographic product* of X, X_1, and Y, respectively.


1
0.8
0.6
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0
0
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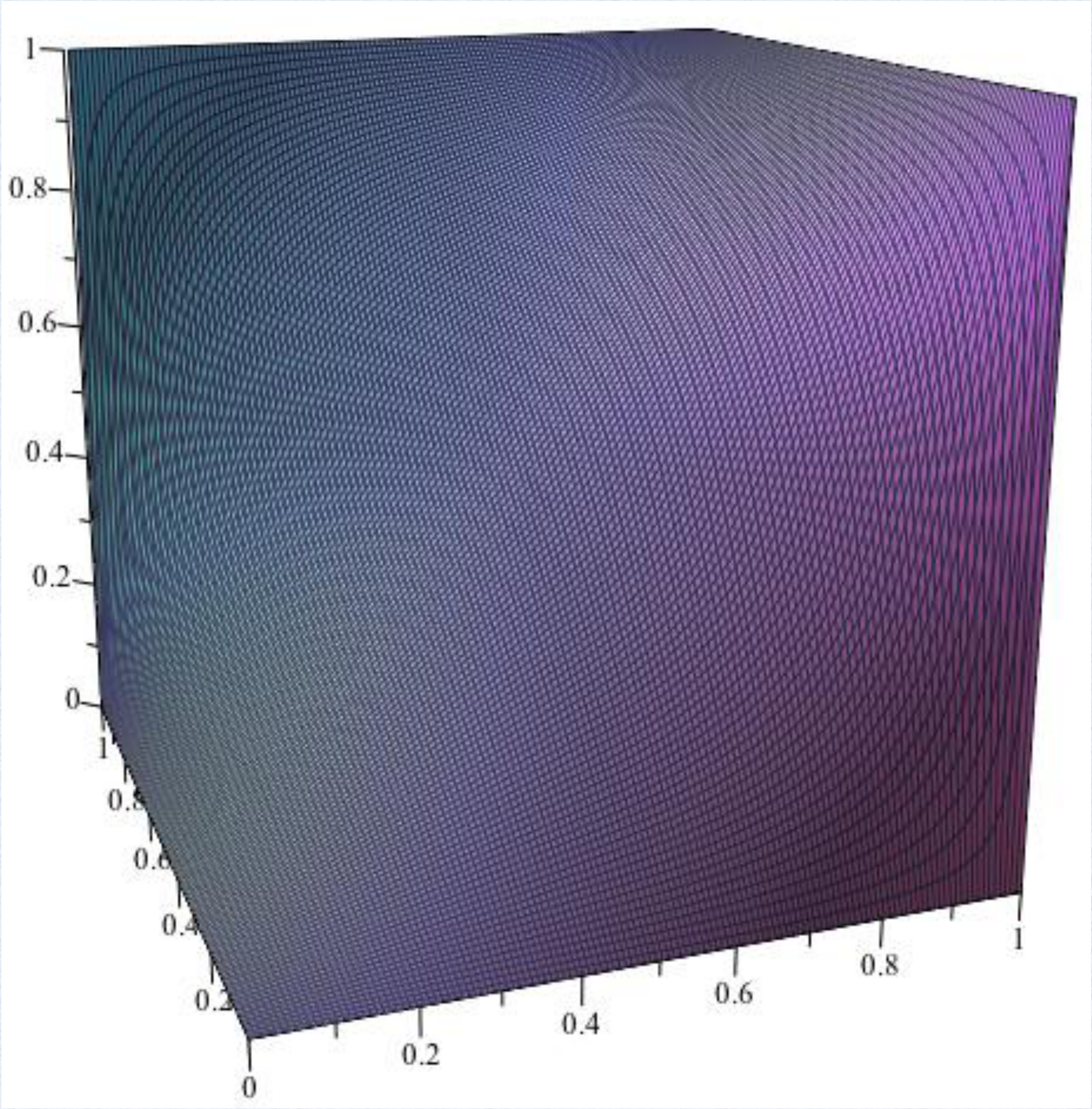
1



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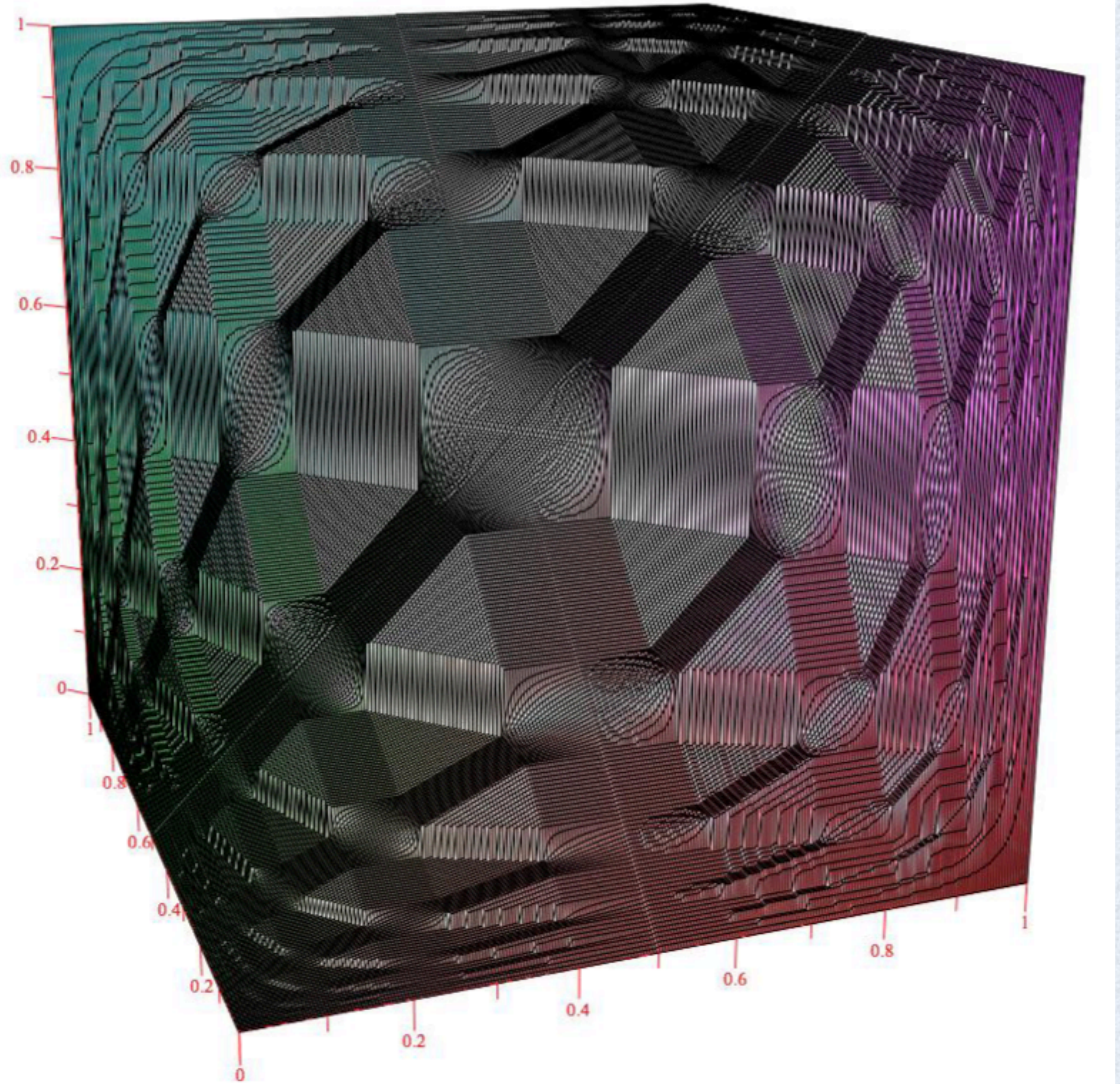
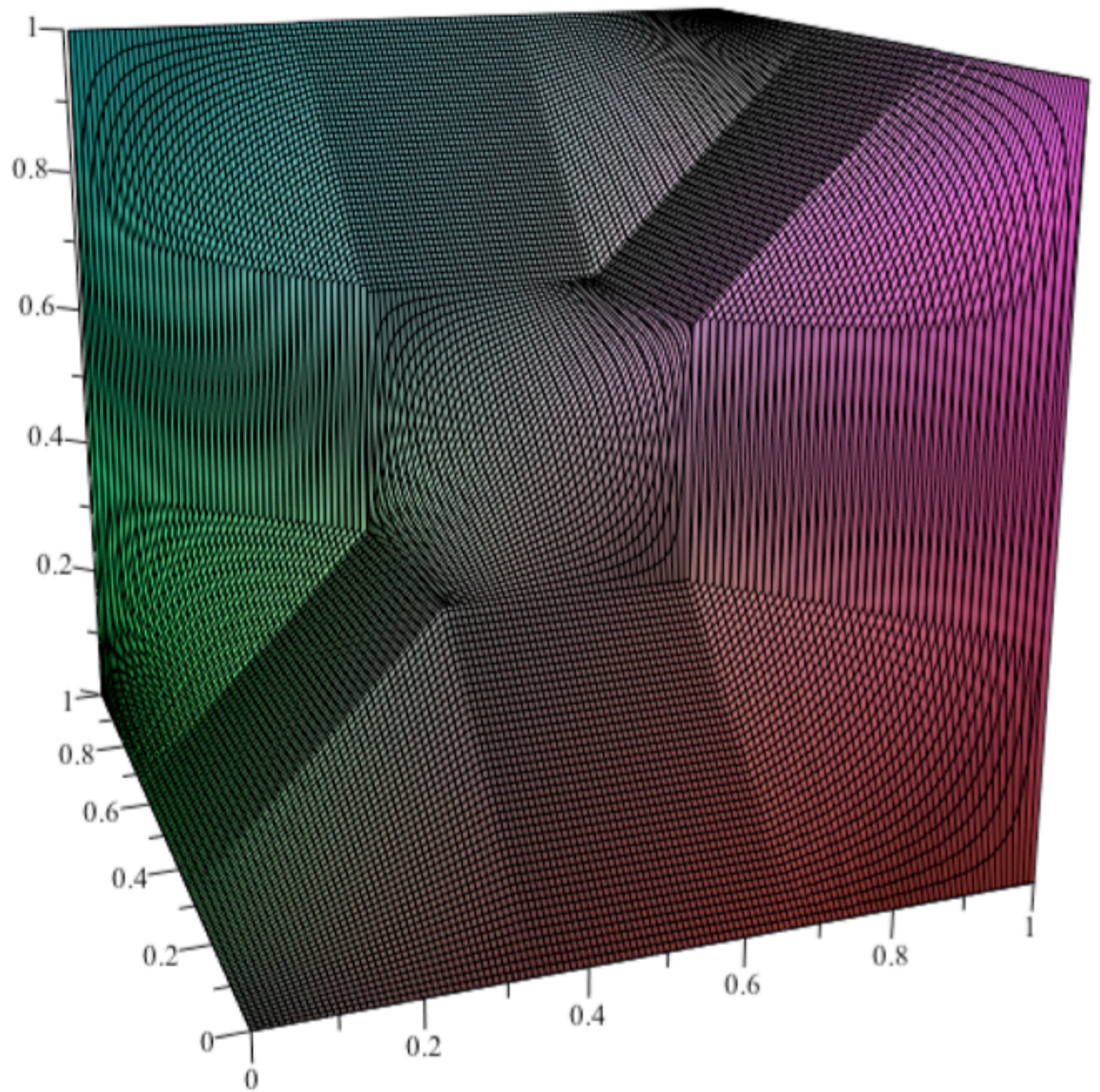
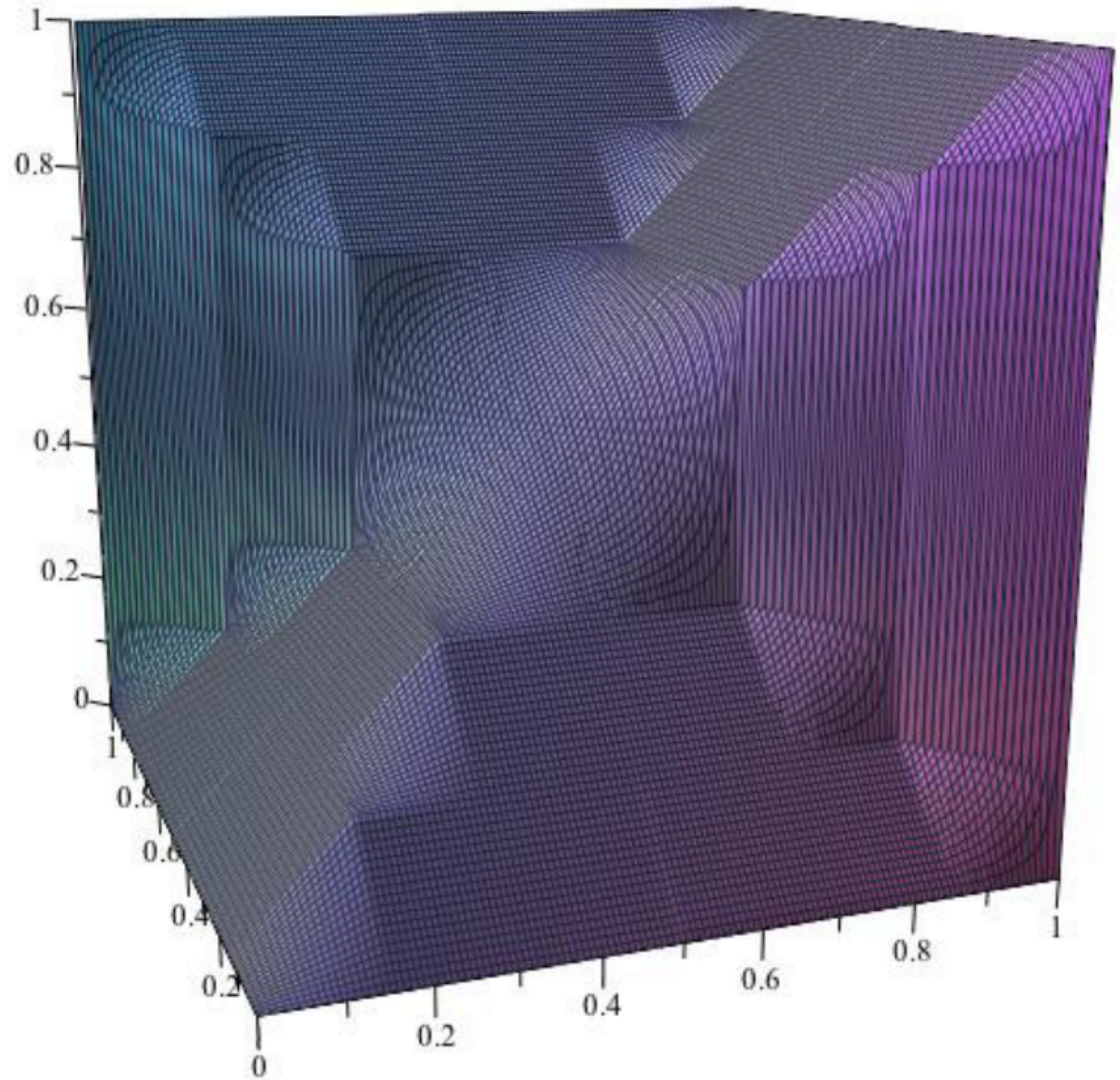


Figure 8: $\mathbf{R}_{\boldsymbol{\Gamma}(\mathbf{N},\mathbf{R}^{\perp\top})}$



$$\mathbf{R}_{\mathbf{\Gamma}(\mathbf{1},\mathbf{R}^{\perp\top})}$$



$$(\mathbf{R}_{\mathbf{\Gamma}(\mathbf{1},\mathbf{R}^{\perp\top})})_{\mathbf{\Gamma}(\mathbf{1},\mathbf{R}^{\perp\top})}$$

Definition 1. *(Partial-lexicographic products)*
Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be a group-like FL_e-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \star, \to_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $'^*$ and $'^\star$, respectively.

Add a top element $\top$ to Y, and extend $\star$ by $\top \star y = y \star \top = \top$ for $y \in Y \cup \{\top\}$, then add a bottom element $\perp$ to $Y \cup \{\top\}$, and extend $\star$ by $\perp \star y = y \star \perp = \perp$ for $y \in Y \cup \{\perp, \top\}$.

Let $\mathbf{X_1} = (X_1, \wedge_X, \vee_X, *, \to_*, t_X, f_X)$ be any cancellative subalgebra of $\mathbf{X}$ (by Theorem 1, $\mathbf{X_1}$ is a lattice ordered group). We define

$$\mathbf{X}_{\mathbf{\Gamma(X_1,Y^{\perp\top})}} = \left(X_{\Gamma(X_1,Y^{\perp\top})}, \leq, \circledast, \to_\circledast, (t_X, t_Y), (f_X, f_Y)\right),$$

where

$$X_{\Gamma(X_1,Y^{\perp\top})} = (X_1 \times (Y \cup \{\perp, \top\})) \cup ((X \setminus X_1) \times \{\perp\}),$$

$\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y \cup \{\perp,\top\}}$ to $X_{\Gamma(X_1,Y^{\perp\top})}$, $\circledast$ is defined coordinatewise, and the operation $\to_\circledast$ is given by $(x_1, y_1) \to_\circledast (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ where

$$(x, y)' = \begin{cases} (x'^*, y'^\star) & \text{if } x \in X_1 \\ (x'^*, \perp) & \text{if } x \notin X_1 \end{cases}.$$

Call $\mathbf{X}_{\mathbf{\Gamma(X_1,Y^{\perp\top})}}$ the *(type-I) partial-lexicographic product* of X, X_1, and Y, respectively.

Let $\mathbf{X} = (X, \leq_X, *, \to_*, t_X, f_X)$ be a group-like FL_e-chain, $\mathbf{Y} = (Y, \leq_Y, \star, \to_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $'^*$ and $'^\star$, respectively.

Add a top element $\top$ to Y, and extend $\star$ by $\top \star y = y \star \top = \top$ for $y \in Y \cup \{\top\}$.

Further, let $\mathbf{X_1} = (X_1, \wedge, \vee, *, \to_*, t_X, f_X)$ be a cancellative, discrete, prime[1] subalgebra of $\mathbf{X}$ (by Theorem 1, $\mathbf{X_1}$ is a discrete lattice ordered group). We define

$$\mathbf{X}_{\mathbf{\Gamma(X_1,Y^{\top})}} = \left(X_{\Gamma(X_1,Y^{\top})}, \leq, \circledast, \to_\circledast, (t_X, t_Y), (f_X, f_Y)\right),$$

where

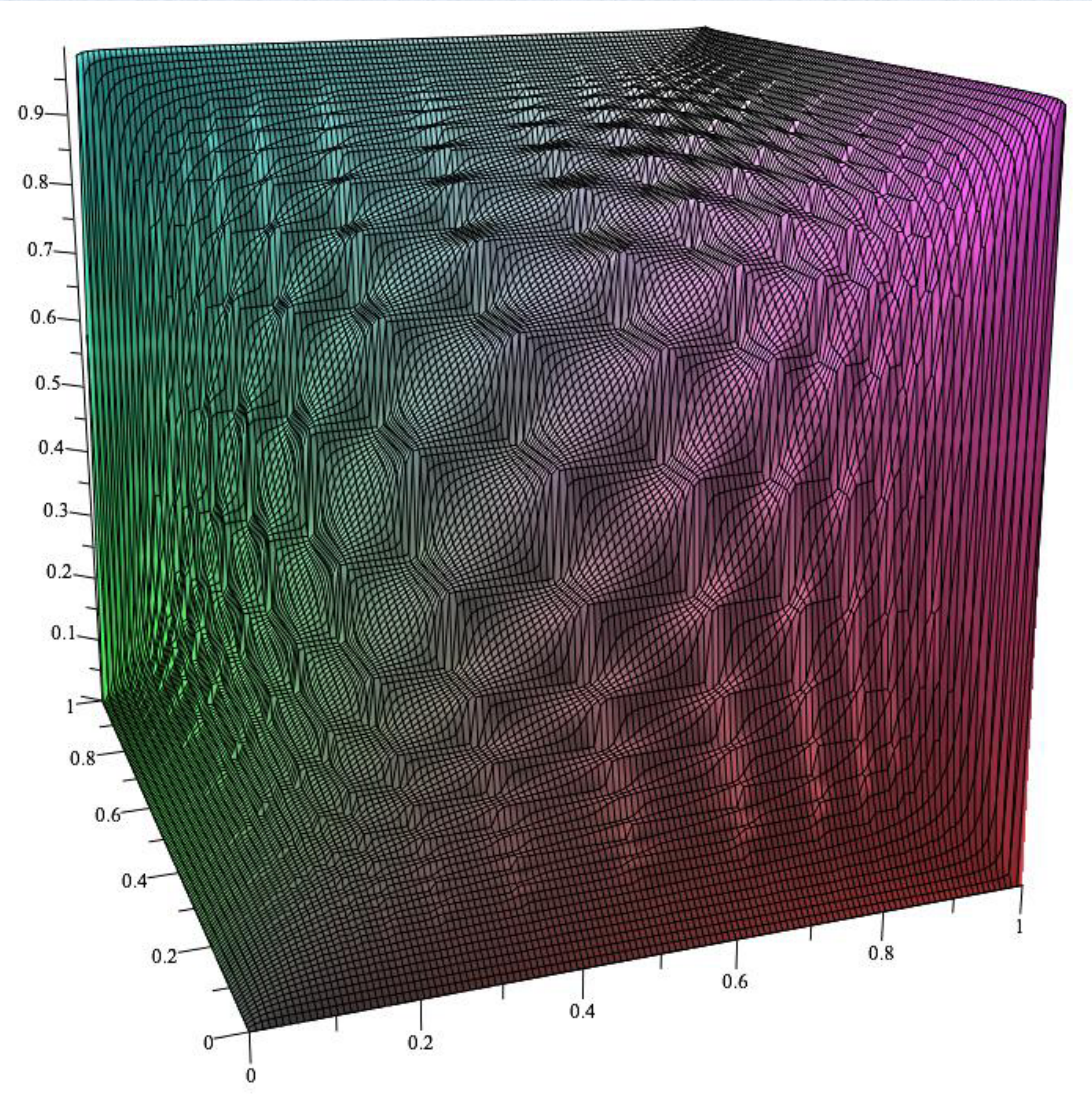
$$X_{\Gamma(X_1,Y^{\top})} = (X_1 \times (Y \cup \{\top\})) \cup ((X \setminus X_1) \times \{\top\}),$$

$\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y \cup \{\top\}}$ to $X_{\Gamma(X_1,Y)}$, $\circledast$ is defined coordinatewise, and the operation $\to_\circledast$ is given by $(x_1, y_1) \to_\circledast (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ where

$$(x, y)' = \begin{cases} ((x'^*), \top) & \text{if } x \notin X_1 \text{ and } y = \top \\ (x'^*, y'^\star) & \text{if } x \in X_1 \text{ and } y \in Y \\ ((x'^*)_\downarrow, \top) & \text{if } x \in X_1 \text{ and } y = \top \end{cases}.$$

$$x_\downarrow = \begin{cases} u & \text{if there exists } u < x \text{ such that there is no element in } X \text{ between } u \text{ and } x, \\ x & \text{if for any } u < x \text{ there exists } v \in X \text{ such that } u < v < x. \end{cases}$$

Call $\mathbf{X}_{\mathbf{\Gamma(X_1,Y^{\top})}}$ the *(type-II) partial-lexicographic product* of X, X_1, and Y, respectively.



$\mathbf{N}_{\mathbf{\Gamma}(\mathbf{N},\mathbf{R}^{\top})}$

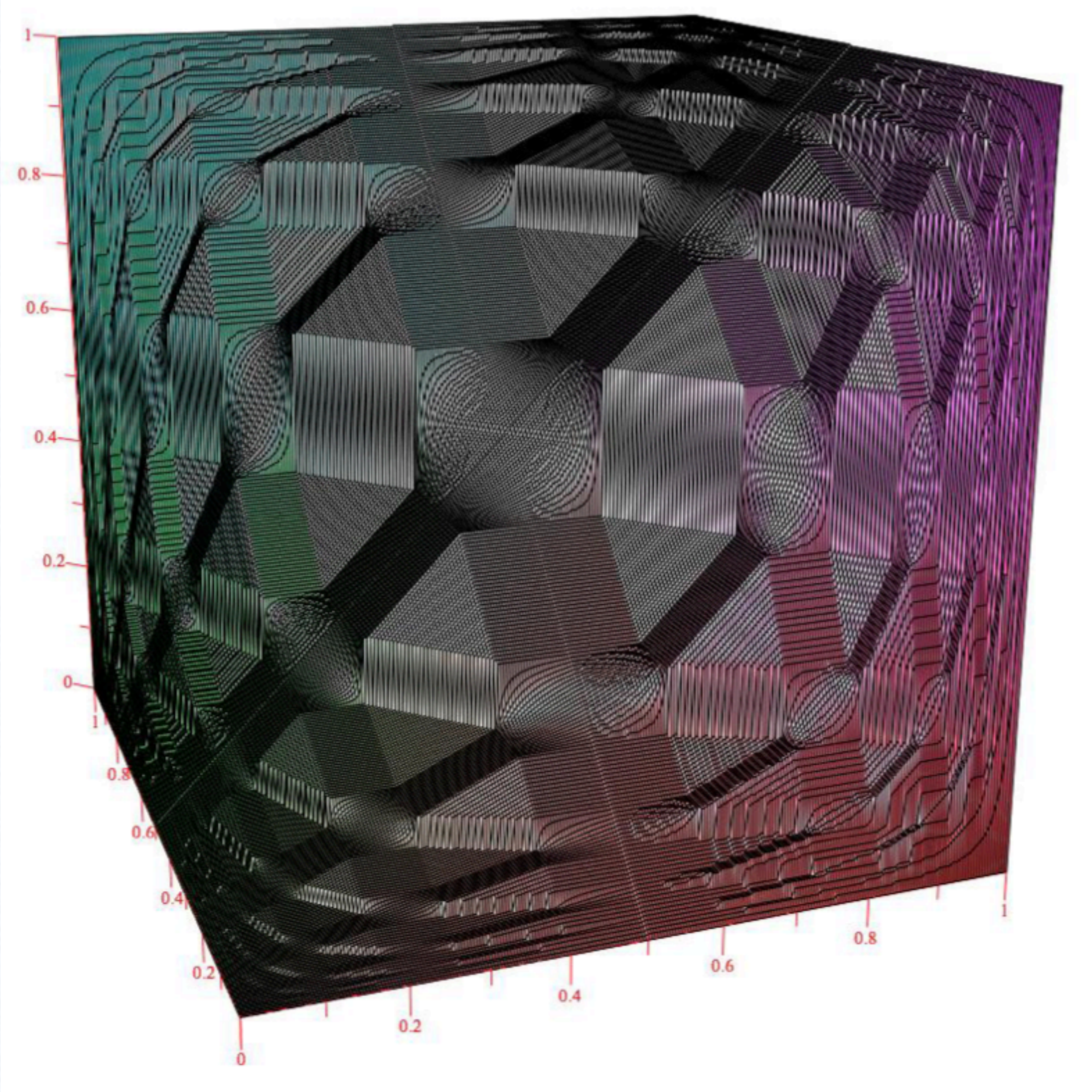
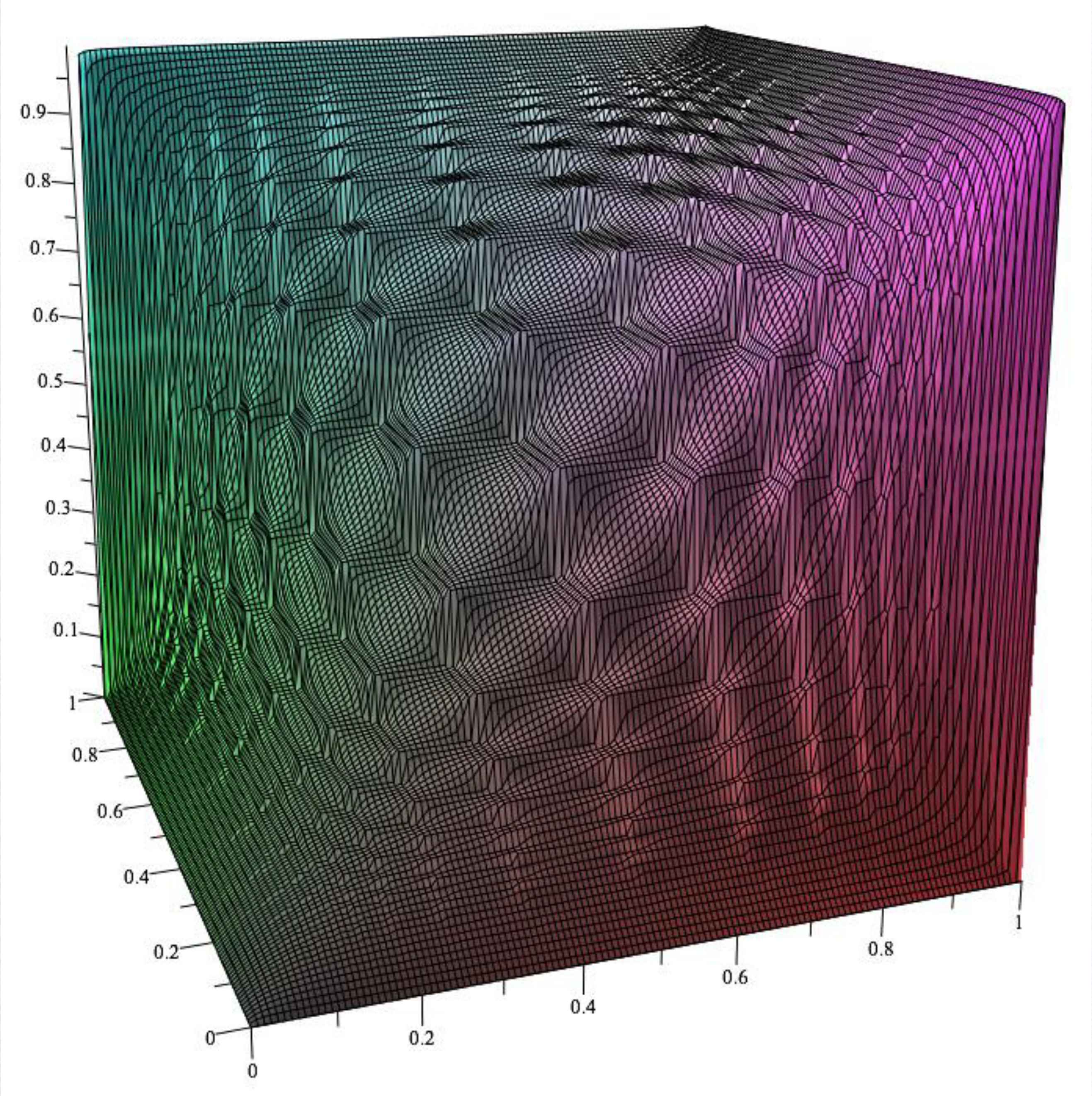


Figure 8: $\mathbf{R}_{\boldsymbol{\Gamma}(\mathbf{N},\mathbf{R}^{\perp\top})}$



$\mathbf{N}_{\boldsymbol{\Gamma}(\mathbf{N},\mathbf{R}^{\top})}$

Theorem 2. *$\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1},\mathbf{Y}^{\perp\top})}$ and $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1},\mathbf{Y}^{\top})}$ are involutive FL_e-algebras. If $\mathbf{Y}$ is group-like then also $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1},\mathbf{Y}^{\perp\top})}$ and $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1},\mathbf{Y}^{\top})}$ are group-like.*

Main Result

Representation by totally ordered Abelian Groups

Theorem 2.20. (Structural description) *If* $\mathbf{X}$ *is a densely ordered, group-like* FL_e*-chain, which has only* $n \in \mathbf{N}$ *idempotents in its positive cone then there exist linearly ordered Abelian groups* $\mathbf{G}_i$ $(i \in \{1, 2, \ldots, n\})$, $\mathbf{H}_1 \leq \mathbf{G}_1$, $\mathbf{H}_i \leq \mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i)$ $(i \in \{2, \ldots, n-1\})$, *and a binary sequence* $\iota \in \{\top\bot, \top\}^{\{2,\ldots,n\}}$ *such that* $\mathbf{X} \simeq \mathbf{X}_n$, *where* $\mathbf{X}_1 := \mathbf{G}_1$ *and* $\mathbf{X}_i := \mathbf{X}_{i-1}{}_{\mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i{}^{\iota_i})}$ $(i \in \{2, \ldots, n\})$. [24]

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[24] In the spirit of Theorem 2.5 we identify linearly ordered Abelian groups by cancellative, group-like FL_e-chains here; the isomorphism is meant between FL_e-algebras. Read $\leq$ as 'subgroup'.

Surprising?

- Every commutative integral monoid on a finite chain is an FL_{ew}-chain.
- It has been shown in [SJ, F Montagna, A Proof of Standard Completeness for Esteva and Godo's Logic MTL, STUDIA LOGICA 70:(2) pp. 183-192. (2002)] that any FL_{ew}-chain embeds into a densely-ordered FL_{ew}-chain.
- By the rotation construction [18, Theorem 3], any densely-ordered FL_{ew}-chain embeds into a densely-ordered, involutive FL_{ew}-chain.
- FL_e-chains, with the additionally postulated $t = f$ condition and with the assumption on the number of idempotent elements results in a such a strong structural representation, which uses only linearly ordered Abelian groups.

Embedding

Corollary 2.22. (Hahn-type embedding) *Densely ordered, group-like FL_e-chains with a finite number of idempotents embed in the partial-lexicographic product of real groups.*

Corollary 2.23. (Lexicographical embedding of the monoid reduct) *The monoid reduct of a densely ordered, group-like FL_e-chain with a finite number of idempotents embed in the* lexicographic *product of extended real lines*[26].

Standard completeness of IUL?

(plus t <--> f)

Densely-ordered group-like FL_e-chains (with finitely many idempotents)

That is all!

$$\tau(x) = x \to_{\circledast} x$$

Theorem 2.5. *For a group-like FL_e-algebra $(X, \wedge, \vee, \circledast, \to_{\circledast}, t, f)$ the following statements are equivalent:*

(1) *Each element of X has inverse given by $x^{-1} = x'$, and hence $(X, \wedge, \vee, \circledast, t)$ is a lattice-ordered Abelian group,*
(2) *$\circledast$ is cancellative,*
(3) *$\tau(x) = t$ for all $x \in X$.*
(4) *The only idempotent element in the positive cone of X is t.*

Definition 2.12. For a group-like FL_e-chain $(X, \wedge, \vee, \circledast, \rightarrow_{\circledast}, t, f)$, for $u \geq t$ and $\Box \in \{<, =, \geq\}$ denote

$$X_{\tau \,\Box\, u} \quad = \quad \{x \in X : \tau(x) \Box u\}.$$

(1) $X_{\tau<u} \cup \{t\}$, $X_{\tau=u} \cup \{t\}$, $X_{\tau\geq u} \cup \{t\}$ *are nonempty subuniverses.*

Definition 2.14. Let $(X, \leq, \circledast, \rightarrow_{\circledast}, t, f)$ be a group-like FL_e-chain, $u \geq t$ idempotent, and $\Box \in \{<, =, \geq\}$. For $x, y \in X_{\tau\Box u}$, define $x \sim_{\Box} y$ if $z \in X_{\tau\Box u}$ holds for any $z \in X$, $x < z < y$. It is an equivalence relation on $X_{\tau\Box u}$ since the order is linear. Denote the component of x by $[x]_{\tau\Box u}$ and call it the convex component of x with respect to $\tau\Box u$. If u and $\Box$ are clear from the context we shall simply write $[x]$. Define

$$X_{[\tau\Box u]} = \{[x]_{\tau\Box u} : x \in X_{\tau\Box u}\}.$$

Definition 2.16. Let $\mathbf{X} = (X, \leq, \circledast, \rightarrow_{\circledast}, t, f)$ be a group-like FL_e-chain. Let $u > t$ be idempotent. For $x \in X_{\tau<u}$ let $\top_{[x]} := \bigvee_{z\in[x]} z$ and $\bot_{[x]} := \bigwedge_{z\in[x]} z$.

$$\overline{[x]}_{\tau<u} \quad = \quad [x]_{\tau<u} \cup \{\bot_{[x]}, \top_{[x]}\}$$

$$\widetilde{[x]}_{\tau<u} \quad = \quad [x]_{\tau<u} \cup \{\top_{[x]}\}$$

Lemma 2.20. **(Decomposition - Type I and II)** *Let $\mathbf{X} = (X, \leq, \circledast, \rightarrow_{\circledast}, t, f)$ be a densely-ordered, group-like FL_e-chain. Let $u > t$ be idempotent.*

(1) *Assume that u' is idempotent.*

 (a) $\bar{\mathbf{X}}_{[\tau<u]} = (\bar{X}_{[\tau<u]}, \leq_{\star}, \star, \overline{[t]})$ *is a linearly ordered Abelian group with inverse operation* $\overset{\star}{\prime}$.

 (b) $\mathbf{X}_{\bar{\mathbf{u}}} = (\bar{X}_{[\tau<u]} \cup \dot{X}_{\tau\geq u}, \leq_{\star}, \star, \rightarrow_{\star}, \overline{[t]}, \overline{[f]})$ *is a group-like FL_e-chain with involution* $\overset{\star}{\prime}$ *and $\bar{\mathbf{X}}_{[\tau<u]}$* (qua *group-like FL_e-chain) is a cancellative subalgebra of $\mathbf{X}_{\bar{\mathbf{u}}}$. $\mathbf{X}_{\bar{\mathbf{u}}}$ is densely-ordered and the set of positive idempotents of $\mathbf{X}_{\bar{\mathbf{u}}}$ is order-isomorphic to the set of positive idempotents of $\mathbf{X}$ deprived of u.*

 (c) *If u is the smallest idempotent above t then*

$$\mathbf{X} \simeq (\mathbf{X}_{\bar{\mathbf{u}}})_{\mathbf{\Gamma}(\bar{\mathbf{X}}_{[\tau<u]}, [\mathbf{t}]_{[\tau<u]}{}^{\top\perp})}.$$

(2) *Assume that u' is not idempotent.*

 (a) $\widetilde{\mathbf{X}}_{[\tau<u]} = (\widetilde{X}_{[\tau<u]}, \leq_{*}, *, \widetilde{[t]})$ *is a linearly ordered Abelian group with inverse operation* $\overset{*}{\prime}$.

 (b) $\mathbf{X}_{\widetilde{\mathbf{u}}} = (\widetilde{X}_{[\tau<u]} \cup X^{\circ}_{\tau\geq u}, \leq_{*}, *, \rightarrow_{*}, \widetilde{[t]}, \widetilde{[f]})$ *is a group-like FL_e-chain with involution* $\overset{*}{\prime}$, *and $\widetilde{\mathbf{X}}_{[\tau<u]}$* (qua *group-like FL_e-chain) is a cancellative, discrete, prime subalgebra of $\mathbf{X}_{\widetilde{\mathbf{u}}}$. $\mathbf{X}_{\widetilde{\mathbf{u}}}$ is densely-ordered, and the set of positive idempotents of $\mathbf{X}_{\widetilde{\mathbf{u}}}$ is order-isomorphic to the set of positive idempotents of $\mathbf{X}$ deprived of u.*

 (c) *If u is the smallest idempotent above t then*

$$\mathbf{X} \simeq (\mathbf{X}_{\widetilde{\mathbf{u}}})_{\mathbf{\Gamma}(\widetilde{\mathbf{X}}_{[\tau<u]}, [\mathbf{t}]_{[\tau<u]}{}^{\top})}.$$

Theorem 2.21. **(Structural representation)** *If* $\mathbf{X}$ *is a densely-ordered, group-like* FL_e*-chain, which has only* $n \in \mathbf{N}$ *idempotents in its positive cone then there exist linearly ordered Abelian groups* $\mathbf{G}_i$ $(i \in \{1, 2, \ldots, n\})$, $\mathbf{H}_1 \leq \mathbf{G}_1$, $\mathbf{H}_i \leq \mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i)$ $(i \in \{2, \ldots, n-1\})$*, and a binary sequence* $\iota \in \{\top\bot, \top\}^{\{2,\ldots,n\}}$ *such that* $\mathbf{X} \simeq \mathbf{X}_n$*, where* $\mathbf{X}_1 := \mathbf{G}_1$ *and* $\mathbf{X}_i := {\mathbf{X}_{i-1}}_{\mathbf{\Gamma}(\mathbf{H}_{i-1}, {\mathbf{G}_i}^{\iota_i})}$ $(i \in \{2, \ldots, n\})$. [32]

[32] In the spirit of Theorem 2.5 we identify linearly ordered Abelian groups by cancellative, group-like FL_e-chains here; the isomorphism is meant between FL_e-algebras.

That is really all!