On two concepts of ultrafilter extensions of first-order models and their generalizations

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Part I: Ultrafilter extensions

Space of ultrafilters

For X a set, let $\beta X = {\mathfrak{u} : \mathfrak{u} \text{ is an ultrafilter over } X}$. Basic open sets: $\widetilde{S} = {\mathfrak{u} \in \beta X : S \in \mathfrak{u}}$ for all $S \subseteq X$.

Facts. The space βX is:

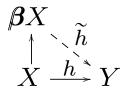
(i) compact,

(ii) Hausdorff,

(iii) extremally disconnected,

(iv) includes X as a dense subspace (identifying X with the set of principal ultrafilters),

(v) the largest compactification of X endowed with the discrete topology: For any compact Hausdorff Y and any $h : X \to Y$, there is a unique continuous extension $\tilde{h} : \beta X \to Y$:



Ultrafilter extensions of models

Fix a first-order language and consider an arbitrary model

$$\mathfrak{A} = (X, F, \dots, R, \dots)$$

with the universe X, operations F, \ldots , and relations R, \ldots on X.

An abstract *ultrafilter extension* of \mathfrak{A} is a model \mathfrak{A}' (in the same language) of form

$$\mathfrak{A}' = (\boldsymbol{\beta} X, F', \dots, R', \dots)$$

with the universe βX in which operations F', \ldots and relations R', \ldots extend F, \ldots and R, \ldots resp.

There are essentially two known canonical ways to extend relations by ultrafilters, and only one to extend maps.

Extending relations

Let $R \subseteq X_1 \times \ldots \times X_n$.

Larger extension

Define $R^* \subseteq \beta X_1 \times \ldots \times \beta X_n$ by letting

$$R^*(\mathfrak{u}_1,\ldots,\mathfrak{u}_n) \text{ iff}$$
$$(\forall A_1 \in \mathfrak{u}_1) \ldots (\forall A_n \in \mathfrak{u}_n) (\exists x_1 \in A_1) \ldots (\exists x_n \in A_n)$$
$$R(x_1,\ldots,x_n).$$

Smaller extension

Define $\widetilde{R} \subseteq \beta X_1 \times \ldots \times \beta X_n$ by letting

 $\widetilde{R}(\mathfrak{u}_1,\ldots,\mathfrak{u}_n) \text{ iff} \\ \left\{ x_1 \in X_1 : \ldots \{ x_n \in X_n : R(x_1,\ldots,x_n) \} \in \mathfrak{u}_n \ldots \right\} \in \mathfrak{u}_1.$

Ultrafilter quantifiers

Let $(\forall^{\mathfrak{u}} x) \varphi(x, \ldots)$ means $\{x : \varphi(x, \ldots)\} \in \mathfrak{u}$.

Facts.

1. Self-dual: $\forall^{\mathfrak{u}}$ and $\exists^{\mathfrak{u}}$ are equivalent.

2. Do not commute: $(\forall \ \mathfrak{u} x)(\forall \ \mathfrak{v} y)$ and $(\forall \ \mathfrak{v} y)(\forall \ \mathfrak{u} x)$ are not equivalent.

3. Second-order: $(\forall^{\mathfrak{u}}x)$ is equivalent to $(\forall A \in \mathfrak{u})(\exists x \in A)$ and also to $(\exists A \in \mathfrak{u})(\forall x \in A)$.

Rewritting smaller extensions of relations

Via ultrafilter quantifiers:

 $\widetilde{R}(\mathfrak{u}_1,\ldots,\mathfrak{u}_n)$ iff $(\forall^{\mathfrak{u}_1}x_1)\ldots(\forall^{\mathfrak{u}_n}x_n) R(x_1,\ldots,x_n).$

Via second-order quantifiers:

$$\begin{split} \widetilde{R}(\mathfrak{u}_1,\ldots,\mathfrak{u}_n) & \text{iff} \\ (\forall A_1 \in \mathfrak{u}_1)(\exists x_1 \in A_1)\ldots(\forall A_n \in \mathfrak{u}_n)(\exists x_n \in A_n) \\ & R(x_1,\ldots,x_n). \end{split}$$

Facts. 1. If R is unary, then $\widetilde{R} = R^*$ is basic open. In general, $\widetilde{R} \subseteq R^*$.

2. The extensions vs operations on relations:

	_	\cap	\cup	0	-1
\sim	1	1	1	0	0
	0	0	1	1	1

An opposite character: \sim well behaves with Boolean but not "group-like" operations, while * conversely.

Extending maps

Let $F : X_1 \times \ldots \times X_n \to Y$. The extended map $\widetilde{F} : \beta X_1 \times \ldots \times \beta X_n \to \beta Y$ is defined by letting

$$\widetilde{F}(\mathfrak{u}_1,\ldots,\mathfrak{u}_n) = \{A \subseteq Y : \{x_1 \in X_1 : \ldots \{x_n \in X_n : F(x_1,\ldots,x_n) \in A\} \in \mathfrak{u}_n \ldots\} \in \mathfrak{u}_1 \}.$$

Rewritting via ultrafilter quantifiers:

$$\widetilde{F}(\mathfrak{u}_1,\ldots,\mathfrak{u}_n) = \{A \subseteq Y : (\forall^{\mathfrak{u}_1}x_1)\ldots(\forall^{\mathfrak{u}_n}x_n) F(x_1,\ldots,x_n) \in A\}.$$

Facts. Let F be a unary map. Then the map \widetilde{F} : (i) is continuous,

(ii) coincides with R^* (but not with \widetilde{R}) where R is F considered as a binary relation.

Both items are not true for maps of bigger arity.

Extending models

Let $\mathfrak{A} = (\beta X, F, \dots, R, \dots)$ be a first-order model. Define two ultrafilter extensions of \mathfrak{A} as follows:

Larger extension

$$\mathfrak{A}^* = (\boldsymbol{\beta} X, \widetilde{F}, \dots, R^*, \dots).$$

Smaller extension

$$\widetilde{\mathfrak{A}} = (\boldsymbol{\beta} X, \widetilde{F}, \dots, \widetilde{R}, \dots).$$

Both are canonical in a sense explained below.

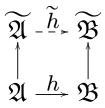
First Extension Theorem

Theorem. Let h be a homomorphism between models \mathfrak{A} and \mathfrak{B} . The continuous extension \tilde{h} is:

(i) a homomorphism between \mathfrak{A}^* and \mathfrak{B}^* :

$$\mathfrak{A}^{*} \xrightarrow{\widetilde{h}} \mathfrak{B}^{*}$$

(ii) a homomorphism between $\widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}}$:



Both (i) and (ii) remain true for embeddings and some other model-theoretic interrelations.

This is a partial case of the much stronger Second Extension Theorem. To formulate it, describe topological properties of both extensions.

Topology of the extensions

Let X_1, \ldots, X_n, Y be topological spaces and suppose $A_1 \subseteq X_1, \ldots, A_{n-1} \subseteq X_{n-1}$.

1. A map $F: X_1 \times \ldots \times X_n \to Y$ is right continuous w.r.t. A_1, \ldots, A_{n-1} iff for each $i, 1 \leq i \leq n$, and every $a_1 \in A_1, \ldots, a_{i-1} \in A_{i-1}$ and $x_{i+1} \in X_{i+1}, \ldots, x_n \in X_n$, the map

$$x \mapsto F(a_1, \ldots, a_{i-1}, x, x_{i+1}, \ldots, x_n)$$

of X_i into Y is continuous.

2. A relation $R \subseteq X_1 \times \ldots \times X_n$ is right open (right closed, etc.) w.r.t. A_1, \ldots, A_{n-1} iff for each $i, 1 \leq i \leq n$, and every $a_1 \in A_1, \ldots, a_{i-1} \in A_{i-1}$ and $x_{i+1} \in X_{i+1}, \ldots, x_n \in X_n$, the set

$$\left\{x\in X_i: R(a_1,\ldots,a_{i-1},x,x_{i+1},\ldots,x_n)\right\}$$
 is open (closed, etc.) in $X_i.$

Theorem. If $\mathfrak{A} = (A, F, \dots, R, \dots)$ is a model then:

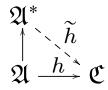
(i) all operations \widetilde{F}, \ldots in the extensions \mathfrak{A}^* and $\widetilde{\mathfrak{A}}$ are right continuous w.r.t. A, (ii) all relations \widetilde{R}, \ldots in the extension $\widetilde{\mathfrak{A}}$ are right clopen w.r.t. A, (iii) all relations R^*, \ldots in the extension \mathfrak{A}^* are closed in the product topology.

This allows us to consider models whose topological properties are similar to the properties of ultrafilter extensions of each of the described two types.

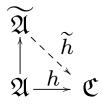
Second Extension Theorem

Theorem. Let \mathfrak{A} and \mathfrak{C} be two models, h a homomorphism of \mathfrak{A} into \mathfrak{C} , and let \mathfrak{C} carry a compact Hausdorff topology in which all operations are right continuous w.r.t. h "A (the image of the universe of \mathfrak{A} under h).

(i) If all relations in \mathfrak{C} are closed, then \tilde{h} is a homomorphism of \mathfrak{A}^* into \mathfrak{C} :



(ii) If all relations in \mathfrak{C} are right closed w.r.t. $h^{"}A$, then \tilde{h} is a homomorphism of \mathfrak{A} into \mathfrak{C} :



(Note: in (ii), the class of target models \mathfrak{C} is wider.)

Facts. 1. The First Extension Theorem follows from the Second one: pick \mathfrak{C} equal to \mathfrak{B}^* , resp., to $\widetilde{\mathfrak{B}}$.

2. The Second Extension Theorem generalizes the classical fact about the space βX as the largest compactification of a discrete space X to the case when X carries an arbitrary first-order structure.

3. The models \mathfrak{A}^* and $\widetilde{\mathfrak{A}}$ are unique (up to isomorphism) extensions of \mathfrak{A} satisfying the theorem.

2. and 3. shows that both extensions can be considered as canonical.

Historical remarks

Largest compactification:

Tychonoff spaces: Čech, Stone (indep., 1937), T_1 -spaces: Wallman (1938).

Larger extension of relations:

Jónsson and Tarski (1951, 1952), rediscovered by Lemmon and Scott (1966), Goldblatt and Thomason (1975), more explicitly: Goldblatt (1989). The name "ultrafilter extension": probably van Benthem (1979).

Smaller extension of relations:

Saveliev (2011).

Extension of maps:

Pairing: Kochen (1961), Frayne, Morel, and Scott (1963) (then Gaifman, Kunen, and many others for iterated ultraproducts).

Multiplication in semigroups: Galvin and Glazer (1974) (then Hindman and many others for algebra of ul-trafilters).

In general: Goranko (2007) and Saveliev (2011).

The First Extension theorem:

Larger extension: Goranko (2007), smaller extension: Saveliev (2011).

The Second Extension Theorem:

Larger extension: Saveliev (2014), smaller extension: Saveliev (2011).

Part II: Generalized models

Extension of extension

Immediate purpose. An alternative description of the larger extension of relation using continuous extensions of maps. For this, we extend the extension procedure *itself*.

For functions $f: X \to Y$, let

$$\operatorname{ext}(f) = \widetilde{f}.$$

Then:

(i) ext is a map of Y^X into $C(\beta X, \beta Y)$,

(ii) $C(\beta X, \beta Y)$ with the standard (pointwise convergence) topology is a compact Hausdorff space, hence:

(iii) ext continuously extends to $\widetilde{\text{ext}}$ on $\beta(Y^X)$:

$$\begin{array}{c} \boldsymbol{\beta}(Y^X) \\ \uparrow \quad \widehat{\boldsymbol{\nabla}} \\ Y^X \xrightarrow{\boldsymbol{\mathsf{ext}}} \\ C(\boldsymbol{\beta}X, \boldsymbol{\beta}Y) \end{array}$$

The map $\widetilde{\text{ext}}$ is surjective and non-injective.

If X = n, then $\beta X = n$ and so

$$C(\beta X, \beta Y) = (\beta Y)^n = \beta Y \times \ldots \times \beta Y$$
 (*n* times).

The alternative description of the *-extension:

Theorem. Let $R \subseteq X \times \ldots \times X$. Then

$$R^* = \widetilde{\mathsf{ext}} \, `` \mathsf{cl}_{\beta(X^n)} R,$$

the image of the closure of R in the space $\beta(X^n)$ under $\widetilde{\text{ext}}$.

Using ultrafilters over maps leads to the following idea.

Generalized models

1. A generalized (or ultrafilter) interpretation is a map i that takes:

(i) each n-ary functional symbol F to an ultrafilter over the set of n-ary operations on X,

(ii) each n-ary predicate symbol R to an ultrafilter over the set of n-ary relations on X:

$$i(F) \in \boldsymbol{\beta}(X^{X \times \dots \times X}), \quad i(R) \in \boldsymbol{\beta} \mathcal{P}(X \times \dots \times X).$$

2. An *ultrafilter valuation* of variables is a map v that takes each variable x to an ultrafilter over X:

$$v(x) \in \beta X.$$

3. A generalized model is $(\beta X, \iota(F), \ldots, \iota(R), \ldots)$.

We are going to define the satisfiability relation \models in generalized models.

Valuation of terms

Let app : $X_1 \times \ldots \times X_n \times Y^{X_1 \times \ldots \times X_n} \rightarrow Y$ be the *application* operation:

$$app(a_1, ..., a_n, f) = f(a_1, ..., a_n).$$

Extend it to the map app right continuous w.r.t. the principal ultrafilters, in the usual way:

Given i and v, define v_i on terms by induction: (i) v_i coincides with v on variables, (ii) if v_i has been already defined on terms t_1, \ldots, t_n ,

$$v_i(F(t_1,\ldots,t_n)) = \widetilde{\operatorname{app}}(v_i(t_1),\ldots,v_i(t_n),i(F)).$$

Satisfiability

Let in $\subseteq X_1 \times \ldots \times X_n \times \mathcal{P}(X_1 \times \ldots \times X_n)$ be the *membership* relation:

in
$$(a_1, \ldots, a_n, R)$$
 iff $(a_1, \ldots, a_n) \in R$.

Extend it to the relation

$$\widetilde{\mathsf{in}} \subseteq \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \times \boldsymbol{\beta} \mathcal{P}(X_1 \times \ldots \times X_n)$$

right clopen w.r.t. principal ultrafilters.

Define \models by induction: (i) $\mathfrak{A} \models t_1 = t_2 [v]$ iff $v_i(t_1) = v_i(t_2)$, (ii) if $R(t_1, \ldots, t_n)$ is an atomic formula in which R is not the equality predicate,

 $\mathfrak{A} \models R(t_1, \ldots, t_n) [v] \text{ iff } \widetilde{\mathrm{in}}(v_i(t_1), \ldots, v_i(t_n), i(P)),$

(iii) if $\varphi(t_1, \ldots, t_n)$ is obtained by Boolean connectives or quantifiers from formulas for which \models has been already defined, define $\mathfrak{A} \models \varphi[v]$ in the standard way.

Remarks.

Generalized models generalize not all ordinary models but those that are ultrafilter extensions of some models.

If a generalized interpretation is *principal* (all nonlogical symbols are interpreted by principal ultrafilters), it is identified with an ordinary interpretation with the same universe βX . Not every ordinary interpretation with the universe βX is of this form.

Precise relationships between generalized models, ordinary models, and ultrafilter extensions will be described below. For this, we provide two operations, e and E, which turn generalized models into certain ordinary models that generalize *- and \sim -extensions.

Operation e

First expand the domain of ext by *n*-ary functions: for $f: X_1 \times \ldots \times X_n \to Y$, let

$$\mathsf{ext}(f) = \widetilde{f}.$$

Then:

(i) ext is a map of $Y^{X_1 \times \ldots \times X_n}$ into

$$RC_{X_1,\ldots,X_{n-1}}(\beta X_1 \times \ldots \times \beta X_n, \beta Y),$$

the set of functions of $\beta X_1 \times \ldots \times \beta X_n$ into βY that are right continuous w.r.t. X_1, \ldots, X_{n-1} ,

(ii) the latter set is closed in the compact Hausdorff space of all functions

$$\beta Y^{\beta X_1 \times \ldots \times \beta X_n}$$

so is compact Hausdorff too,

(iii) ext continuously extends to $\widetilde{\text{ext}}$ on $\beta(Y^{X_1 \times \ldots \times X_n})$.

Define e as follows:

(i) on functions, let e be ext in this expanded meaning, so e takes ultrafilters over functions to functions over ultrafilters:

$$e: \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n}) \to \boldsymbol{\beta}Y^{\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n},$$

(ii) by identifying relations with their characteristic functions, let also e take ultrafilters over relations to relations over ultrafilters:

$$e: \boldsymbol{\beta} \mathcal{P}(X_1 \times \ldots \times X_n) \to \mathcal{P}(\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n).$$

Fact. Both *e* and \widetilde{app} (or \widetilde{in}) are expressed via each other: if $\mathfrak{f} \in \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n})$, $\mathfrak{r} \in \boldsymbol{\beta}\mathcal{P}(X_1 \times \ldots \times X_n)$, and $\mathfrak{u}_1 \in \boldsymbol{\beta}X_1, \ldots, \mathfrak{u}_n \in \boldsymbol{\beta}X_n$, then

$$e(\mathfrak{f})(\mathfrak{u}_1,\ldots,\mathfrak{u}_n) = \widetilde{\operatorname{app}}(\mathfrak{u}_1,\ldots,\mathfrak{u}_n,\mathfrak{f}),$$
$$e(\mathfrak{r})(\mathfrak{u}_1,\ldots,\mathfrak{u}_n) \text{ iff } \widetilde{\operatorname{in}}(\mathfrak{u}_1,\ldots,\mathfrak{u}_n,\mathfrak{r}).$$

For a generalized model $\mathfrak{B} = (\boldsymbol{\beta} X, \mathfrak{f}, \dots, \mathfrak{r}, \dots)$, let

$$e(\mathfrak{B}) = (\boldsymbol{\beta} X, e(\mathfrak{f}), \dots, e(\mathfrak{r}), \dots).$$

Note that $e(\mathfrak{B})$ is an ordinary model with the same universe.

Theorem. Let \mathfrak{A} is a generalized model. Then for all formulas φ and elements $\mathfrak{u}_1, \ldots, \mathfrak{u}_n$ of the universe of \mathfrak{A} ,

 $\mathfrak{A} \models \varphi[\mathfrak{u}_1, \ldots, \mathfrak{u}_n] \text{ iff } e(\mathfrak{A}) \models \varphi[\mathfrak{u}_1, \ldots, \mathfrak{u}_n].$

Operation E

Define a map E, with the same domain and range that the map e has, as follows:

(i) E and e coincide on $\beta(Y^{X_1 \times \ldots \times X_n})$, (ii) if $\mathfrak{r} \in \beta \mathcal{P}(X_1 \times \ldots \times X_n)$ then $E(\mathfrak{r}) = \{ \widetilde{\mathsf{ext}}(\mathfrak{q}) : \mathfrak{q} \in \widetilde{\mathsf{ext}}(\mathfrak{r}) \}.$

Fact. $E(\mathfrak{r})$ is a closed subspace of $\beta X_1 \times \ldots \times \beta X_n$.

Proposition. Let $\mathfrak{r} \in \beta \mathcal{P}(X_1 \times \ldots \times X_n)$. Then

 $e(\mathfrak{r}) = \widetilde{R}$ and $E(\mathfrak{r}) = R^*$

for $R = e(\mathfrak{r}) \cap (X_1 \times \ldots \times X_n) = E(\mathfrak{r}) \cap (X_1 \times \ldots \times X_n) = \bigcap_{S \in \mathfrak{r}} \bigcup S.$

For a generalized model $\mathfrak{B} = (\boldsymbol{\beta} X, \mathfrak{f}, \dots, \mathfrak{r}, \dots)$, let

$$E(\mathfrak{B}) = (\boldsymbol{\beta}X, E(\mathfrak{f}), \dots, E(\mathfrak{r}), \dots).$$

Then $E(\mathfrak{B})$, like $e(\mathfrak{B})$, is an ordinary model with the same universe.

By Proposition above, whether the models $e(\mathfrak{B})$ and $E(\mathfrak{B})$ are ultrafilter extensions of some models depends only on the generalized interpretation of functional symbols in \mathfrak{B} .

Generalized models vs ultrafilter extensions

An ultrafilter f over functions is *pseudo-principal* iff app takes any tuple consisting of principal ultrafilters together with f to a principal ultrafilter:

 $a_1 \in X_1, \ldots, a_n \in X_n$ implies $\widetilde{\operatorname{app}}(a_1, \ldots, a_n, \mathfrak{f}) \in Y$.

Facts. 1. Every principal \mathfrak{f} is pseudo-principal. 2. There exist pseudo-principal ultrafilters that are not principal as well as ultrafilters that are not pseudoprincipal.

A generalized interpretation i is *pseudo-principal on* functional symbols iff i(F) is a pseudo-principal ultrafilter for each functional symbol F. **Theorem.** Let \mathfrak{B} be a generalized model with the universe βX . The following are equivalent:

(i) $e(\mathfrak{B}) = \widetilde{\mathfrak{A}}$ for a model \mathfrak{A} with the universe X, (ii) $E(\mathfrak{B}) = \mathfrak{A}^*$ for a model \mathfrak{A} with the universe X, (iii) the interpretation in \mathfrak{B} is pseudo-principal on functional symbols.

Moreover, the model \mathfrak{A} in (i) and (ii) is the same.

Generalized models vs ordinary models

Whether an ordinary model with the universe βX is of form $e(\mathfrak{B})$ or $E(\mathfrak{B})$, for some generalized model \mathfrak{B} (clearly, with the same universe βX) depends only on its topological properties:

Theorem. Let \mathfrak{A} be an ordinary model with the universe βX . Then:

(i) $\mathfrak{A} = e(\mathfrak{B})$ for a generalized model \mathfrak{B} iff in \mathfrak{A} all operations are right continuous w.r.t. X and all relations are right clopen w.r.t. X, (ii) $\mathfrak{A} = E(\mathfrak{B})$ for a generalized model \mathfrak{B} iff in \mathfrak{A} all operations are right continuous w.r.t. X and all relations are closed.

Note: two last theorem together generalizes the description of topological properties of ultrafilter extensions from Part I.