On two concepts of ultrafilter extensions of first-order models and their generalizations

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Part I: Ultrafilter extensions

Space of ultrafilters

For X a set, let $\boldsymbol{\beta}X = \{\mathfrak{u} : \mathfrak{u}$ is an ultrafilter over $X\}$.
Basic open sets: $\widetilde{S} = \{\mathfrak{u} \in \boldsymbol{\beta}X : S \in \mathfrak{u}\}$ for all $S \subseteq X$.

Facts. The space $\boldsymbol{\beta}X$ is:
(i) compact,
(ii) Hausdorff,
(iii) extremally disconnected,
(iv) includes X as a dense subspace (identifying X with the set of principal ultrafilters),
(v) the largest compactification of X endowed with the discrete topology: *For any compact Hausdorff Y and any $h : X \to Y$, there is a unique continuous extension $\widetilde{h} : \boldsymbol{\beta}X \to Y$:*

$$\begin{tikzcd}
\boldsymbol{\beta}X \arrow[dr, dashed, "\widetilde{h}"] & \\
X \arrow[u] \arrow[r, "h"'] & Y
\end{tikzcd}$$

Ultrafilter extensions of models

Fix a first-order language and consider an arbitrary model

$$\mathfrak{A} = (X, F, \ldots, R, \ldots)$$

with the universe X, operations $F, \ldots$, and relations $R, \ldots$ on X.

An abstract *ultrafilter extension* of $\mathfrak{A}$ is a model $\mathfrak{A}'$ (in the same language) of form

$$\mathfrak{A}' = (\boldsymbol{\beta} X, F', \ldots, R', \ldots)$$

with the universe $\boldsymbol{\beta} X$ in which operations $F', \ldots$ and relations $R', \ldots$ extend $F, \ldots$ and $R, \ldots$ resp.

There are essentially two known canonical ways to extend relations by ultrafilters, and only one to extend maps.

Extending relations

Let $R \subseteq X_1 \times \ldots \times X_n$.

Larger extension

Define $R^* \subseteq \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n$ by letting

$$R^*(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \text{ iff}$$
$$(\forall A_1 \in \mathfrak{u}_1) \ldots (\forall A_n \in \mathfrak{u}_n)(\exists x_1 \in A_1) \ldots (\exists x_n \in A_n)$$
$$R(x_1, \ldots, x_n).$$

Smaller extension

Define $\widetilde{R} \subseteq \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n$ by letting

$$\widetilde{R}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \text{ iff}$$
$$\Big\{x_1 \in X_1 : \ldots \{x_n \in X_n : R(x_1, \ldots, x_n)\} \in \mathfrak{u}_n \ldots\Big\} \in \mathfrak{u}_1.$$

Ultrafilter quantifiers

Let $(\forall^{\mathfrak{u}} x)\, \varphi(x, \ldots)$ means $\{x : \varphi(x, \ldots)\} \in \mathfrak{u}$.

Facts.

1. Self-dual: $\forall^{\mathfrak{u}}$ and $\exists^{\mathfrak{u}}$ are equivalent.

2. Do not commute: $(\forall^{\mathfrak{u}} x)(\forall^{\mathfrak{v}} y)$ and $(\forall^{\mathfrak{v}} y)(\forall^{\mathfrak{u}} x)$ are not equivalent.

3. Second-order: $(\forall^{\mathfrak{u}} x)$ is equivalent to $(\forall A \in \mathfrak{u})(\exists x \in A)$ and also to $(\exists A \in \mathfrak{u})(\forall x \in A)$.

Rewritting smaller extensions of relations

Via ultrafilter quantifiers:

$$\widetilde{R}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \ \text{ iff } \ (\forall^{\mathfrak{u}_1} x_1) \ldots (\forall^{\mathfrak{u}_n} x_n)\ R(x_1, \ldots, x_n).$$

Via second-order quantifiers:

$$\widetilde{R}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \ \text{ iff }$$
$$(\forall A_1 \in \mathfrak{u}_1)(\exists x_1 \in A_1) \ldots (\forall A_n \in \mathfrak{u}_n)(\exists x_n \in A_n)$$
$$R(x_1, \ldots, x_n).$$

Facts. 1. If R is unary, then $\widetilde{R} = R^*$ is basic open. In general, $\widetilde{R} \subseteq R^*$.

2. The extensions vs operations on relations:

	$-$	$\cap$	$\cup$	$\circ$	$^{-1}$
$\sim$	1	1	1	0	0
$*$	0	0	1	1	1

An opposite character: $^\sim$ well behaves with Boolean but not "group-like" operations, while * conversely.

Extending maps

Let $F : X_1 \times \ldots \times X_n \to Y$. The extended map $\widetilde{F} : \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \to \boldsymbol{\beta} Y$ is defined by letting

$$\widetilde{F}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) =$$
$$\Big\{A \subseteq Y : \{x_1 \in X_1 : \ldots \{x_n \in X_n : F(x_1, \ldots, x_n) \in A\} \in \mathfrak{u}_n \ldots\} \in \mathfrak{u}_1\Big\}.$$

Rewritting via ultrafilter quantifiers:

$$\widetilde{F}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) =$$
$$\Big\{A \subseteq Y : (\forall^{\mathfrak{u}_1} x_1) \ldots (\forall^{\mathfrak{u}_n} x_n)\ F(x_1, \ldots, x_n) \in A\Big\}.$$

Facts. Let F be a unary map. Then the map $\widetilde{F}$:
(i) is continuous,
(ii) coincides with R^* (but not with $\widetilde{R}$) where R is F considered as a binary relation.
Both items are not true for maps of bigger arity.

Extending models

Let $\mathfrak{A} = (\boldsymbol{\beta} X, F, \ldots, R, \ldots)$ be a first-order model. Define two ultrafilter extensions of $\mathfrak{A}$ as follows:

Larger extension

$$\mathfrak{A}^* = (\boldsymbol{\beta} X, \widetilde{F}, \ldots, R^*, \ldots).$$

Smaller extension

$$\widetilde{\mathfrak{A}} = (\boldsymbol{\beta} X, \widetilde{F}, \ldots, \widetilde{R}, \ldots).$$

Both are canonical in a sense explained below.

First Extension Theorem

Theorem. *Let h be a homomorphism between models $\mathfrak{A}$ and $\mathfrak{B}$. The continuous extension $\widetilde{h}$ is:*

(i) a homomorphism between $\mathfrak{A}^*$ and $\mathfrak{B}^*$:

$$\begin{tikzcd} \mathfrak{A}^* \arrow[r, dashed, "\widetilde{h}"] & \mathfrak{B}^* \\ \mathfrak{A} \arrow[u] \arrow[r, "h"] & \mathfrak{B} \arrow[u] \end{tikzcd}$$

(ii) a homomorphism between $\widetilde{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}}$:

$$\begin{tikzcd} \widetilde{\mathfrak{A}} \arrow[r, dashed, "\widetilde{h}"] & \widetilde{\mathfrak{B}} \\ \mathfrak{A} \arrow[u] \arrow[r, "h"] & \mathfrak{B} \arrow[u] \end{tikzcd}$$

Both (i) and (ii) remain true for embeddings and some other model-theoretic interrelations.

This is a partial case of the much stronger Second Extension Theorem. To formulate it, describe topological properties of both extensions.

Topology of the extensions

Let $X_1, \ldots, X_n, Y$ be topological spaces and suppose $A_1 \subseteq X_1, \ldots, A_{n-1} \subseteq X_{n-1}$.

1. A map $F : X_1 \times \ldots \times X_n \to Y$ is *right continuous w.r.t.* $A_1, \ldots, A_{n-1}$ iff for each i, $1 \leqslant i \leqslant n$, and every $a_1 \in A_1, \ldots, a_{i-1} \in A_{i-1}$ and $x_{i+1} \in X_{i+1}, \ldots, x_n \in X_n$, the map

$$x \mapsto F(a_1, \ldots, a_{i-1}, x, x_{i+1}, \ldots, x_n)$$

of X_i into Y is continuous.

2. A relation $R \subseteq X_1 \times \ldots \times X_n$ is *right open* (*right closed*, etc.) *w.r.t.* $A_1, \ldots, A_{n-1}$ iff for each i, $1 \leqslant i \leqslant n$, and every $a_1 \in A_1, \ldots, a_{i-1} \in A_{i-1}$ and $x_{i+1} \in X_{i+1}, \ldots, x_n \in X_n$, the set

$$\{x \in X_i : R(a_1, \ldots, a_{i-1}, x, x_{i+1}, \ldots, x_n)\}$$

is open (closed, etc.) in X_i.

Theorem. *If* $\mathfrak{A} = (A, F, \ldots, R, \ldots)$ *is a model then:*

(i) all operations $\widetilde{F}, \ldots$ *in the extensions* $\mathfrak{A}^*$ *and* $\widetilde{\mathfrak{A}}$ *are right continuous w.r.t.* A*,*
(ii) all relations $\widetilde{R}, \ldots$ *in the extension* $\widetilde{\mathfrak{A}}$ *are right clopen w.r.t.* A*,*
(iii) all relations $R^*, \ldots$ *in the extension* $\mathfrak{A}^*$ *are closed in the product topology.*

This allows us to consider models whose topological properties are similar to the properties of ultrafilter extensions of each of the described two types.

Second Extension Theorem

Theorem. *Let $\mathfrak{A}$ and $\mathfrak{C}$ be two models, h a homomorphism of $\mathfrak{A}$ into $\mathfrak{C}$, and let $\mathfrak{C}$ carry a compact Hausdorff topology in which all operations are right continuous w.r.t. $h``A$ (the image of the universe of $\mathfrak{A}$ under h).*

(i) If all relations in $\mathfrak{C}$ are closed, then $\widetilde{h}$ is a homomorphism of $\mathfrak{A}^*$ into $\mathfrak{C}$:

$$\begin{tikzcd} \mathfrak{A}^* \arrow[dr, dashed, "\widetilde{h}"] & \\ \mathfrak{A} \arrow[u] \arrow[r, "h"] & \mathfrak{C} \end{tikzcd}$$

(ii) If all relations in $\mathfrak{C}$ are right closed w.r.t. $h``A$, then $\widetilde{h}$ is a homomorphism of $\widetilde{\mathfrak{A}}$ into $\mathfrak{C}$:

$$\begin{tikzcd} \widetilde{\mathfrak{A}} \arrow[dr, dashed, "\widetilde{h}"] & \\ \mathfrak{A} \arrow[u] \arrow[r, "h"] & \mathfrak{C} \end{tikzcd}$$

(Note: in (ii), the class of target models $\mathfrak{C}$ is wider.)

Facts. 1. The First Extension Theorem follows from the Second one: pick $\mathfrak{C}$ equal to $\mathfrak{B}^*$, resp., to $\widetilde{\mathfrak{B}}$.

2. The Second Extension Theorem generalizes the classical fact about the space $\boldsymbol{\beta}X$ as the largest compactification of a discrete space X to the case when X carries an arbitrary first-order structure.

3. The models $\mathfrak{A}^*$ and $\widetilde{\mathfrak{A}}$ are unique (up to isomorphism) extensions of $\mathfrak{A}$ satisfying the theorem.

2. and 3. shows that both extensions can be considered as canonical.

Historical remarks

Largest compactification:

Tychonoff spaces: Čech, Stone (indep., 1937), T_1-spaces: Wallman (1938).

Larger extension of relations:

Jónsson and Tarski (1951, 1952), rediscovered by Lemmon and Scott (1966), Goldblatt and Thomason (1975), more explicitly: Goldblatt (1989). The name "ultrafilter extension": probably van Benthem (1979).

Smaller extension of relations:

Saveliev (2011).

Extension of maps:

Pairing: Kochen (1961), Frayne, Morel, and Scott (1963) (then Gaifman, Kunen, and many others for iterated ultraproducts).
Multiplication in semigroups: Galvin and Glazer (1974) (then Hindman and many others for algebra of ultrafilters).
In general: Goranko (2007) and Saveliev (2011).

The First Extension theorem:

Larger extension: Goranko (2007), smaller extension: Saveliev (2011).

The Second Extension Theorem:

Larger extension: Saveliev (2014), smaller extension: Saveliev (2011).

Part II: Generalized models

Extension of extension

Immediate purpose. An alternative description of the larger extension of relation using continuous extensions of maps. For this, we extend the extension procedure *itself*.

For functions $f : X \to Y$, let

$$\mathrm{ext}(f) = \widetilde{f}.$$

Then:

(i) ext is a map of Y^X into $C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)$,
(ii) $C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)$ with the standard (pointwise convergence) topology is a compact Hausdorff space, hence:
(iii) ext continuously extends to $\widetilde{\mathrm{ext}}$ on $\boldsymbol{\beta}(Y^X)$:

$$\begin{tikzcd}
\boldsymbol{\beta}(Y^X) \arrow[dr, dashed, "\widetilde{\mathrm{ext}}"] & \\
Y^X \arrow[u] \arrow[r, "\mathrm{ext}"] & C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y)
\end{tikzcd}$$

The map $\widetilde{\mathrm{ext}}$ is surjective and non-injective.

If $X = n$, then $\boldsymbol{\beta} X = n$ and so

$$C(\boldsymbol{\beta} X, \boldsymbol{\beta} Y) = (\boldsymbol{\beta} Y)^n = \boldsymbol{\beta} Y \times \ldots \times \boldsymbol{\beta} Y \ \ (n \text{ times}).$$

The alternative description of the *-extension:

Theorem. *Let* $R \subseteq X \times \ldots \times X$. *Then*

$$R^* = \widetilde{\mathsf{ext}}\,\text{“}\,\mathsf{cl}_{\boldsymbol{\beta}(X^n)} R,$$

the image of the closure of R *in the space* $\boldsymbol{\beta}(X^n)$ *under* $\widetilde{\mathsf{ext}}$.

Using ultrafilters over maps leads to the following idea.

Generalized models

1. A *generalized* (or *ultrafilter*) *interpretation* is a map $\imath$ that takes:
(i) each n-ary functional symbol F to an ultrafilter over the set of n-ary operations on X,
(ii) each n-ary predicate symbol R to an ultrafilter over the set of n-ary relations on X:

$$\imath(F) \in \boldsymbol{\beta}(X^{X \times \ldots \times X}), \quad \imath(R) \in \boldsymbol{\beta}\,\mathcal{P}(X \times \ldots \times X).$$

2. An *ultrafilter valuation* of variables is a map v that takes each variable x to an ultrafilter over X:

$$v(x) \in \boldsymbol{\beta}X.$$

3. A *generalized model* is $(\boldsymbol{\beta}X, \imath(F), \ldots, \imath(R), \ldots)$.

We are going to define the satisfiability relation $\Vdash$ in generalized models.

Valuation of terms

Let app : $X_1 \times \ldots \times X_n \times Y^{X_1 \times \ldots \times X_n} \to Y$ be the *application* operation:

$$\mathsf{app}(a_1, \ldots, a_n, f) = f(a_1, \ldots, a_n).$$

Extend it to the map $\widetilde{\mathsf{app}}$ right continuous w.r.t. the principal ultrafilters, in the usual way:

$$\begin{tikzcd} \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \times \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n}) \arrow[r, dashed, "\widetilde{\mathsf{app}}"] & \boldsymbol{\beta} Y \\ X_1 \times \ldots \times X_n \times Y^{X_1 \times \ldots \times X_n} \arrow[u] \arrow[r, "\mathsf{app}"] & Y \arrow[u] \end{tikzcd}$$

Given $\imath$ and v, define $v_\imath$ on terms by induction:
(i) $v_\imath$ coincides with v on variables,
(ii) if $v_\imath$ has been already defined on terms $t_1, \ldots, t_n$,

$$v_\imath(F(t_1, \ldots, t_n)) = \widetilde{\mathsf{app}}(v_\imath(t_1), \ldots, v_\imath(t_n), \imath(F)).$$

Satisfiability

Let in $\subseteq X_1 \times \ldots \times X_n \times \mathcal{P}(X_1 \times \ldots \times X_n)$ be the *membership* relation:

$$\mathrm{in}\,(a_1, \ldots, a_n, R) \ \text{ iff } \ (a_1, \ldots, a_n) \in R.$$

Extend it to the relation

$$\widetilde{\mathrm{in}} \subseteq \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \times \boldsymbol{\beta}\,\mathcal{P}(X_1 \times \ldots \times X_n)$$

right clopen w.r.t. principal ultrafilters.

Define $\Vdash$ by induction:
(i) $\mathfrak{A} \Vdash t_1 = t_2\,[v]$ iff $v_\imath(t_1) = v_\imath(t_2)$,
(ii) if $R(t_1, \ldots, t_n)$ is an atomic formula in which R is not the equality predicate,

$$\mathfrak{A} \Vdash R(t_1, \ldots, t_n)\,[v] \ \text{ iff } \ \widetilde{\mathrm{in}}\,(v_\imath(t_1), \ldots, v_\imath(t_n), \imath(P)),$$

(iii) if $\varphi(t_1, \ldots, t_n)$ is obtained by Boolean connectives or quantifiers from formulas for which $\Vdash$ has been already defined, define $\mathfrak{A} \Vdash \varphi\,[v]$ in the standard way.

Remarks.

Generalized models generalize not all ordinary models but those that are ultrafilter extensions of some models.

If a generalized interpretation is *principal* (all nonlogical symbols are interpreted by principal ultrafilters), it is identified with an ordinary interpretation with the same universe $\boldsymbol{\beta} X$. Not every ordinary interpretation with the universe $\boldsymbol{\beta} X$ is of this form.

Precise relationships between generalized models, ordinary models, and ultrafilter extensions will be described below. For this, we provide two operations, e and E, which turn generalized models into certain ordinary models that generalize *- and ${}^\sim$-extensions.

Operation e

First expand the domain of ext by n-ary functions: for $f : X_1 \times \ldots \times X_n \to Y$, let

$$\text{ext}(f) = \widetilde{f}.$$

Then:

(i) ext is a map of $Y^{X_1 \times \ldots \times X_n}$ into

$$RC_{X_1,\ldots,X_{n-1}}(\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n, \boldsymbol{\beta} Y),$$

the set of functions of $\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n$ into $\boldsymbol{\beta} Y$ that are right continuous w.r.t. $X_1, \ldots, X_{n-1}$,
(ii) the latter set is closed in the compact Hausdorff space of all functions

$$\boldsymbol{\beta} Y^{\boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n}$$

so is compact Hausdorff too,
(iii) ext continuously extends to $\widetilde{\text{ext}}$ on $\boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n})$.

Define e as follows:

(i) on functions, let e be $\widetilde{\mathsf{ext}}$ in this expanded meaning, so e takes ultrafilters over functions to functions over ultrafilters:

$$e : \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n}) \to \boldsymbol{\beta}Y^{\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n},$$

(ii) by identifying relations with their characteristic functions, let also e take ultrafilters over relations to relations over ultrafilters:

$$e : \boldsymbol{\beta}\,\mathcal{P}(X_1 \times \ldots \times X_n) \to \mathcal{P}(\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n).$$

Fact. Both e and $\widetilde{\mathsf{app}}$ (or $\widetilde{\mathsf{in}}$) are expressed via each other: if $\mathfrak{f} \in \boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n})$, $\mathfrak{r} \in \boldsymbol{\beta}\,\mathcal{P}(X_1 \times \ldots \times X_n)$, and $\mathfrak{u}_1 \in \boldsymbol{\beta}X_1, \ldots, \mathfrak{u}_n \in \boldsymbol{\beta}X_n$, then

$$e(\mathfrak{f})(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) = \widetilde{\mathsf{app}}(\mathfrak{u}_1, \ldots, \mathfrak{u}_n, \mathfrak{f}),$$
$$e(\mathfrak{r})(\mathfrak{u}_1, \ldots, \mathfrak{u}_n) \;\text{ iff }\; \widetilde{\mathsf{in}}\,(\mathfrak{u}_1, \ldots, \mathfrak{u}_n, \mathfrak{r}).$$

For a generalized model $\mathfrak{B} = (\boldsymbol{\beta} X, \mathfrak{f}, \ldots, \mathfrak{r}, \ldots)$, let

$$e(\mathfrak{B}) = (\boldsymbol{\beta} X, e(\mathfrak{f}), \ldots, e(\mathfrak{r}), \ldots).$$

Note that $e(\mathfrak{B})$ is an ordinary model with the same universe.

Theorem. *Let $\mathfrak{A}$ is a generalized model. Then for all formulas φ and elements $\mathfrak{u}_1, \ldots, \mathfrak{u}_n$ of the universe of $\mathfrak{A}$,*

$$\mathfrak{A} \Vdash \varphi\,[\mathfrak{u}_1, \ldots, \mathfrak{u}_n] \;\; \textit{iff} \;\; e(\mathfrak{A}) \vDash \varphi\,[\mathfrak{u}_1, \ldots, \mathfrak{u}_n].$$

Operation E

Define a map E, with the same domain and range that the map e has, as follows:

(i) E and e coincide on $\boldsymbol{\beta}(Y^{X_1 \times \ldots \times X_n})$,
(ii) if $\mathfrak{r} \in \boldsymbol{\beta}\mathcal{P}(X_1 \times \ldots \times X_n)$ then

$$E(\mathfrak{r}) = \left\{ \widetilde{\mathsf{ext}}(\mathfrak{q}) : \mathfrak{q} \in \widetilde{\mathsf{ext}}(\mathfrak{r}) \right\}.$$

Fact. $E(\mathfrak{r})$ is a closed subspace of $\boldsymbol{\beta}X_1 \times \ldots \times \boldsymbol{\beta}X_n$.

Proposition. *Let* $\mathfrak{r} \in \boldsymbol{\beta}\mathcal{P}(X_1 \times \ldots \times X_n)$*. Then*

$$e(\mathfrak{r}) = \widetilde{R} \text{ and } E(\mathfrak{r}) = R^*$$

for $R = e(\mathfrak{r}) \cap (X_1 \times \ldots \times X_n) = E(\mathfrak{r}) \cap (X_1 \times \ldots \times X_n) = \bigcap_{S \in \mathfrak{r}} \bigcup S$.

For a generalized model $\mathfrak{B} = (\boldsymbol{\beta} X, \mathfrak{f}, \ldots, \mathfrak{r}, \ldots)$, let

$$E(\mathfrak{B}) = (\boldsymbol{\beta} X, E(\mathfrak{f}), \ldots, E(\mathfrak{r}), \ldots).$$

Then $E(\mathfrak{B})$, like $e(\mathfrak{B})$, is an ordinary model with the same universe.

By Proposition above, whether the models $e(\mathfrak{B})$ and $E(\mathfrak{B})$ are ultrafilter extensions of some models depends only on the generalized interpretation of functional symbols in $\mathfrak{B}$.

Generalized models vs ultrafilter extensions

An ultrafilter $\mathfrak{f}$ over functions is *pseudo-principal* iff $\widetilde{\mathsf{app}}$ takes any tuple consisting of principal ultrafilters together with $\mathfrak{f}$ to a principal ultrafilter:

$$a_1 \in X_1, \ldots, a_n \in X_n \text{ implies } \widetilde{\mathsf{app}}(a_1, \ldots, a_n, \mathfrak{f}) \in Y.$$

Facts. 1. Every principal $\mathfrak{f}$ is pseudo-principal.
2. There exist pseudo-principal ultrafilters that are not principal as well as ultrafilters that are not pseudo-principal.

A generalized interpretation $\imath$ is *pseudo-principal on functional symbols* iff $\imath(F)$ is a pseudo-principal ultrafilter for each functional symbol F.

Theorem. *Let $\mathfrak{B}$ be a generalized model with the universe $\boldsymbol{\beta} X$. The following are equivalent:*

(i) $e(\mathfrak{B}) = \widetilde{\mathfrak{A}}$ for a model $\mathfrak{A}$ with the universe X,
(ii) $E(\mathfrak{B}) = \mathfrak{A}^*$ for a model $\mathfrak{A}$ with the universe X,
(iii) the interpretation in $\mathfrak{B}$ is pseudo-principal on functional symbols.

Moreover, the model $\mathfrak{A}$ in (i) and (ii) is the same.

Generalized models vs ordinary models

Whether an ordinary model with the universe $\boldsymbol{\beta}X$ is of form $e(\mathfrak{B})$ or $E(\mathfrak{B})$, for some generalized model $\mathfrak{B}$ (clearly, with the same universe $\boldsymbol{\beta}X$) depends only on its topological properties:

Theorem. *Let $\mathfrak{A}$ be an ordinary model with the universe $\boldsymbol{\beta}X$. Then:*

(i) $\mathfrak{A} = e(\mathfrak{B})$ for a generalized model $\mathfrak{B}$ iff in $\mathfrak{A}$ all operations are right continuous w.r.t. X and all relations are right clopen w.r.t. X,
(ii) $\mathfrak{A} = E(\mathfrak{B})$ for a generalized model $\mathfrak{B}$ iff in $\mathfrak{A}$ all operations are right continuous w.r.t. X and all relations are closed.

Note: two last theorem together generalizes the description of topological properties of ultrafilter extensions from Part I.