Wild Algebras in Cartesian Categorical Logic

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A Conjecture

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A finite-dimensional algebra S over k is "wild" is there is a finitely-generated $(S, k\langle X, Y\rangle)$-bimodule M such that an induced tensor functor

$$M \otimes_{k\langle X,Y\rangle} - \colon k\langle X, Y\rangle\text{-mod} \to S\text{-mod}$$

of categories of finite-dimensional modules reflects isomorphism and preserves and reflects indecomposable modules.

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Conjecture: Every wild finite-dimensional algebra has an undecidable theory of modules [Pre88].

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- Cartesian formulas are defined inductively: atomic formulas and finite conjunctions of cartesian formulas are cartesian. '$x.\exists y \phi$' is cartesian provided that $x, y.\phi$ is cartesian and a certain sequent is provable.
- A regular theory is cartesian if its axioms admit a partial order in which any given axiom is cartesian relative to the subtheory generated by the axioms preceding it in the order.

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- Arrows: classes $[\theta]\colon \{\mathbf{x}.\phi\} \to \{\mathbf{y}.\psi\}$ of $\mathbb{T}$-provably-equivalent cartesian formulas θ that are $\mathbb{T}$-provably functional:

$$\theta \vdash_{\mathbf{x},\mathbf{y}} \phi \wedge \psi$$
$$\theta \wedge \theta[\mathbf{z}/\mathbf{y}] \vdash_{\mathbf{x},\mathbf{y},\mathbf{z}} \mathbf{y} = \mathbf{z}$$
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Theorem

For any cartesian theory $\mathbb{T}$ and any cartesian category $\mathscr{D}$, there is an equivalence

$$\mathbf{Cart}(\mathscr{C}_{\mathbb{T}}, \mathscr{D}) \simeq \mathbb{T}\text{-}\mathrm{Mod}(\mathscr{D}).$$

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- Each function symbol $f\colon A_1, \ldots, A_n \to B$ is interpreted as the morphism

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- Each relation symbol $R \rightarrowtail A_1,\dots,A_n$ is interpreted as a suitable subobject of $\{x_1,\dots x_n.\top\}$.
- A sequent $\phi \vdash \psi$ is provable in $\mathbb{T}$ if, and only if, it is satisfied by $\mathbf{M}_{\mathbb{T}}$.

The Theory of Modules

Let $\mathbb{T}_R$ denote the cartesian theory of R modules over a fixed ring R. This has a single sort A, various function symbols $+\colon A, A \to A$ and $-(-)\colon A \to A$ and $r\colon A \to A$ indexed by $r \in R$ with group axioms

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$$\top \vdash_{x,y,z} ((x+y)+z) = (x+(y+z))$$
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The category of **Set**-models is precisely R-Mod.

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The proof shows that the following statements (1) and (2) are equivalent (much as in [Bau75]).

$$\text{The words } u \text{ and } v \text{ over } \{a, b\} \text{ are equivalent in } N. \tag{1}$$

$$\text{The sequent } \bigwedge_{i=1}^{r}(f_i(x) = g_i(x)) \vdash_x f(x) = g(x) \text{ is provable.} \tag{2}$$

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$$\begin{tikzcd}
\{a, b\}^* \arrow[r, "\rho"] \arrow[d, two heads, "e"'] & \mathscr{C}_{\mathbb{T}} \arrow[d, "\iota"] \\
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which amounts to showing that $\mathscr{C}_{\mathbb{T}}$ interprets the undecidable word problem of N. On the other hand, assuming that (1) fails, the monoid-algebra $k[N]$ can be seen to be a model of $\mathbb{T}_{k\langle X,Y\rangle}$ where (2) fails.

Representation Embeddings

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Definition

A representation embedding of cartesian theories $\mathbb{T}_1 \to \mathbb{T}_2$ is a functor

$$E\colon \mathbb{T}_1\text{-}\mathrm{Mod}(\mathbf{Set}) \to \mathbb{T}_2\text{-}\mathrm{Mod}(\mathbf{Set})$$

that preserves finitely-generated projective models, and that both preserves and reflects epimorphisms when restricted to the full subcategory of finitely-generated projectives.

The Main Theorem

Theorem

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ denote cartesian theories admitting a representation embedding E. There is then a functor $T\colon \mathscr{C}_{\mathbb{T}_1} \to \mathscr{C}_{\mathbb{T}_2}$ that preserves and reflects provability in the sense that $\phi \vdash_{\mathbf{x}} \psi$ is provable in $\mathbb{T}_1$ if, and only if, the image sequent $\phi' \vdash_{\mathbf{y}} \psi'$ associated under T is provable in $\mathbb{T}_2$.

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Lemma

For a cartesian theory $\mathbb{T}$, a cartesian sequent $\phi \vdash_{\mathbf{x}} \psi$ is provable in $\mathbb{T}$ if, and only if, $\{\mathbf{x}.\phi\} \leq \{\mathbf{x}.\psi\}$ holds as subobjects of $\{\mathbf{x}.\top\}$, that is, if, and only if, there is a monic arrow $\{\mathbf{x}.\phi\} \to \{\mathbf{x}.\psi\}$ of $\mathscr{C}_{\mathbb{T}}$.

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This is proved in D1.4 of [Joh01].

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For $\mathscr{C}$ cartesian and Cauchy-complete, the finitely-generated projectives of $\mathbf{Cart}(\mathscr{C}, \mathbf{Set})$ are precisely the representable functors.

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A full proof is in [Lam].

In the diagram Φ and Ψ are equivalences.

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The induced functor $T\colon \mathscr{C}_{\mathbb{T}_1} \to \mathscr{C}_{\mathbb{T}_2}$ preserves and reflects provability by the completeness of cartesian logic and the assumed properties of E.

Corollaries to the Main Theorem

Corollary

If $\mathbb{T}_1$ is undecidable in the sense that there is no algorithm determining whether $\phi \vdash_{\mathsf{x}} \psi$ of $\mathbb{T}_1$ is provable, then $\mathbb{T}_2$ is also undecidable.

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Proof.

The functor $M \otimes_{k\langle X,Y\rangle} -$ is a representation embedding since M is a finitely-generated bimodule and free over $k\langle X, Y\rangle$. □

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In summary, the cartesian theory of $k\langle X, Y\rangle$-module is undecidable. Any wild algebra also has an undecidable cartesian theory of modules. But it is not yet clear that the analogous statements are true for the first-order theories.

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