### Wild Algebras in Cartesian Categorical Logic

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1 / 15

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Introduction: Wild Algebras

Cartesian Logic

An Undecidable Theory

Representation Embeddings

Summary and Conclusion

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Introduction: Wild Algebras

# A Conjecture

Let k denote an algebraically closed field. The free associative algebra  $k\langle X, Y \rangle$  is known to have an undecidable classical first-order theory of modules [Bau75],[Pre88].

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#### Definition

A finite-dimensional algebra S over k is "wild" is there is a finitely-generated (S, k(X, Y))-bimodule M such that an induced tensor functor

$$M \otimes_{k \langle X, Y \rangle} - \colon k \langle X, Y \rangle \operatorname{\!-mod} \to S\operatorname{\!-mod}$$

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Conjecture: Every wild finite-dimensional algebra has an undecidable theory of modules [Pre88].

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- A regular theory is cartesian if its axioms admit a partial order in which any given axiom is cartesian relative to the subtheory generated by the axioms preceding it in the order.

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- Arrows: classes  $[\theta]$ :  $\{\mathbf{x}.\phi\} \rightarrow \{\mathbf{y}.\psi\}$  of  $\mathbb{T}$ -provably-equivalent cartesian formulas  $\theta$  that are  $\mathbb{T}$ -provably functional:

$$\begin{split} \theta \vdash_{\mathbf{x},\mathbf{y}} \phi \wedge \psi \\ \theta \wedge \theta[\mathbf{z}/\mathbf{y}] \vdash_{\mathbf{x},\mathbf{y},\mathbf{z}} \mathbf{y} = \mathbf{z} \\ \phi \vdash_{\mathbf{x}} \exists \mathbf{y} \theta. \end{split}$$

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The syntactic category  $\mathscr{C}_{\mathbb{T}}$  is cartesian, Cauchy-complete, and has a universal property.

#### Theorem

For any cartesian theory  $\mathbb T$  and any cartesian category  $\mathscr D,$  there is an equivalence

$$\mathsf{Cart}(\mathscr{C}_{\mathbb{T}},\mathscr{D})\simeq\mathbb{T} ext{-}\mathrm{Mod}(\mathscr{D}).$$

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- A sequent  $\phi \vdash \psi$  is provable in  $\mathbb{T}$  if, and only if, it is satisfied by  $\mathbf{M}_{\mathbb{T}}$ .

### The Theory of Modules

Let  $\mathbb{T}_R$  denote the cartesian theory of R modules over a fixed ring R. This has a single sort A, various function symbols  $+: A, A \rightarrow A$  and  $-(-): A \rightarrow A$  and  $r: A \rightarrow A$  indexed by  $r \in R$  with group axioms

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$$\top \vdash_{x,y,z} ((x + y) + z) = (x + (y + z))$$
$$\top \vdash_{x,y} (x + y) = (y + x)$$
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$$\top \vdash_{x} rs(x) = r(s(x))$$
  
 
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The category of **Set**-models is precisely *R*-Mod.

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The proof shows that the following statements (1) and (2) are equivalent (much as in [Bau75]).

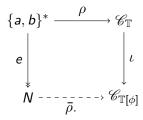
The words u and v over  $\{a, b\}$  are equivalent in N. (1)

The sequent 
$$\bigwedge_{i=1}^{r} (f_i(x) = g_i(x)) \vdash_x f(x) = g(x)$$
 is provable. (2)

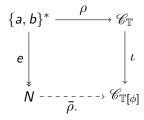
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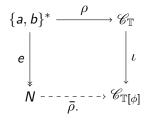


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which amounts to showing that  $\mathscr{C}_{\mathbb{T}}$  interprets the undecidable word problem of *N*. On the other hand, assuming that (1) fails, the monoid-algebra k[N] can be seen to be a model of  $\mathbb{T}_{k\langle X,Y\rangle}$  where (2) fails.

# Representation Embeddings

If  $\mathbb T$  is cartesian, let  $\mathbb T\operatorname{\!-Mod}({\textbf{Set}})$  denote the category of models in  ${\textbf{Set}}.$ 

# Representation Embeddings

If  $\mathbb T$  is cartesian, let  $\mathbb T\operatorname{-Mod}(\mathsf{Set})$  denote the category of models in  $\mathsf{Set}.$  Definition

A representation embedding of cartesian theories  $\mathbb{T}_1 \to \mathbb{T}_2$  is a functor

 $E : \mathbb{T}_1 \operatorname{-Mod}(\operatorname{Set}) \to \mathbb{T}_2 \operatorname{-Mod}(\operatorname{Set})$ 

that preserves finitely-generated projective models, and that both preserves and reflects epimorphisms when restricted to the full subcategory of finitely-generated projectives.

# The Main Theorem

#### Theorem

Let  $\mathbb{T}_1$  and  $\mathbb{T}_2$  denote cartesian theories admitting a representation embedding E. There is then a functor  $T \colon \mathscr{C}_{\mathbb{T}_1} \to \mathscr{C}_{\mathbb{T}_2}$  that preserves and reflects provability in the sense that  $\phi \vdash_{\mathbf{x}} \psi$  is provable in  $\mathbb{T}_1$  if, and only if, the image sequent  $\phi' \vdash_{\mathbf{y}} \psi'$  associated under T is provable in  $\mathbb{T}_2$ .

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For a cartesian theory  $\mathbb{T}$ , a cartesian sequent  $\phi \vdash_{\mathbf{x}} \psi$  is provable in  $\mathbb{T}$  if, and only if,  $\{\mathbf{x}.\phi\} \leq \{\mathbf{x}.\psi\}$  holds as subobjects of  $\{\mathbf{x}.\top\}$ , that is, if, and only if, there is a monic arrow  $\{\mathbf{x}.\phi\} \rightarrow \{\mathbf{x}.\psi\}$  of  $\mathscr{C}_{\mathbb{T}}$ .

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#### Lemma

For  $\mathscr{C}$  cartesian and Cauchy-complete, the finitely-generated projectives of  $Cart(\mathscr{C}, Set)$  are precisely the representable functors.

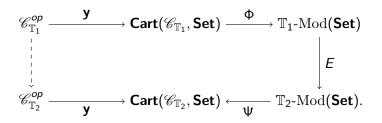
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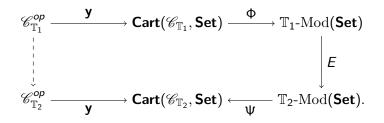
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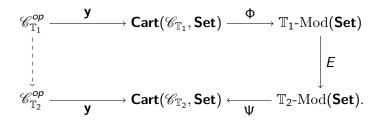


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The induced functor  $T: \mathscr{C}_{\mathbb{T}_1} \to \mathscr{C}_{\mathbb{T}_2}$  preserves and reflects provability by the completeness of cartesian logic and the assumed properties of E.

## Corollaries to the Main Theorem

#### Corollary

If  $\mathbb{T}_1$  is undecidable in the sense that there is no algorithm determining whether  $\phi \vdash_{\mathbf{x}} \psi$  of  $\mathbb{T}_1$  is provable, then  $\mathbb{T}_2$  is also undecidable.

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## Proof.

The functor  $M \otimes_{k\langle X, Y \rangle} -$  is a representation embedding since M is a finitely-generated bimodule and free over  $k\langle X, Y \rangle$ .

Summary and Conclusion

# Does The Theorem Prove the Original Conjecture?

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In summary, the cartesian theory of  $k\langle X, Y \rangle$ -module is undecidable. Any wild algebra also has an undecidable cartesian theory of modules. Summary and Conclusion

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In summary, the cartesian theory of  $k\langle X, Y \rangle$ -module is undecidable. Any wild algebra also has an undecidable cartesian theory of modules. But it is not yet clear that the analogous statements are true for the first-order theories.

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