An approach to parts of d-frames and an Isbell-type density theorem

Imanol Mozo Carollo[1]

imanol.mozo@ehu.es

Chapman University & University of the Basque Country UPV/EHU

eman ta zabal zazu

CHAPMAN UNIVERSITY

Universidad del País Vasco Euskal Herriko Unibertsitatea

[1]Joint work with Andrew Moshier and Joanne Walters-Wayland

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We want have pseudocomplements:

$$\varphi^\bullet := \bigvee\{a \in L_- \mid \varphi \,\mathrm{con}\, a\} \quad \text{such that} \quad \varphi \,\mathrm{con}\, \varphi^\bullet$$

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A *d-frame homomorphism* $f\colon L \to M$ is a pair of frame homomorphisms

$$f_-\colon L_- \to M_- \quad \text{and} \quad f_+\colon L_+ \to M_+$$

that preserves the relations con and tot.

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A. Jung and A. Moshier,
On the bitopological nature of Stone duality,
Technical Report CSR-06-13, The University of Birmingham, 110 pp. (2006).

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J. Picado and A Pultr,
Frames and locales: Topology without points,
Frontiers in Mathematics **28** Springer–Basel (2012).

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Let $f\colon L \to M$ be a d-frame homomorphism. Then

$$f[L] = (f_-(L_-), f_+(L_+), \overline{f(\mathrm{con}_L)}, f(\mathrm{tot}_L)),$$

where $\overline{f(\mathrm{con}_L)}$ is the Scott-closure of $f(\mathrm{con}_L)$, is a d-frame.

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Perception

It is really hard to find examples.

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As extremal epis in **Frm** can be represented uniquely by sublocale sets, we can represent an extremal epi in **dFrm**

$$h\colon L \to M$$

as a pair of sublocales $S_- \in \mathcal{S}(L_-)$ and $S_+ \in \mathcal{S}(L_+)$ endowed with con_S and tot_S relations induced by those of M (as $S_- \simeq M_-$ and $S_+ \simeq M_+$).

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Lemma

In that case,

$$\mathrm{tot}_S = \mathrm{tot}_L \cap S.$$

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We will say that a d-frame homomorphism $f: L \to M$ is *dense* if

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If $f\colon L \to M$ is dense then f_- and f_+ are dense.

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If $f\colon L \to M$ is dense then f_- and f_+ are dense. The reciprocal does not hold.

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An Isbell density theorem

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Let $S_- \in \mathcal{S}(L_-)$ *and* $S_+ \in \mathcal{S}(L_+)$ *such that* $L_-^{\bullet\bullet} \subseteq S_-$ *and* $L_+^{\bullet\bullet} \subseteq S_+$*.*

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Corollary

An extremal epi $h\colon L \to S$ *that is given by sublocales is dense iff* $L_-^{\bullet\bullet} \subseteq S_-$ *and* $L_+^{\bullet\bullet} \subseteq S_+$*.*

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Theorem

Each d-frame has a least dense extremal epi.

Eskerrik asko arretarengatik!

Thank you for your attention!

Děkuji za pozornost!

`imanol.mozo@ehu.eus`

Eskerrik asko arretarengatik!
(This is Basque)

Thank you for your attention!

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(... and I hope this is proper Czech)

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