

# An approach to parts of d-frames and an Isbell-type density theorem

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<sup>1</sup>Joint work with Andrew Moshier and Joanne Walters-Wayland

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We want have pseudocomplements:

$$\varphi^\bullet := \bigvee \{a \in L_- \mid \varphi \text{ con } a\} \quad \text{such that} \quad \varphi \text{ con } \varphi^\bullet$$

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A *d-frame homomorphism*  $f: L \rightarrow M$  is a pair of frame homomorphisms

$$f_-: L_- \rightarrow M_- \quad \text{and} \quad f_+: L_+ \rightarrow M_+$$

that preserves the relations  $\text{con}$  and  $\text{tot}$ .

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A. Jung and A. Moshier,

On the bitopological nature of Stone duality,

*Technical Report CSR-06-13, The University of Birmingham, 110 pp. (2006).*

# Extremal epis in **Frm**



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### Lemma

*Monomorphisms in **Frm** are precisely one-one frame homomorphisms.*

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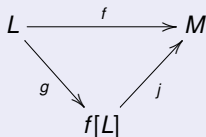
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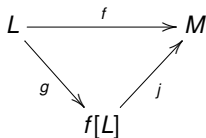
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J. Picado and A Pultr,

Frames and locales: Topology without points,

*Frontiers in Mathematics* **28** Springer–Basel (2012).



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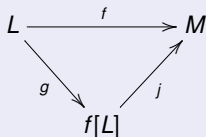
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### Perception

It is really hard to find examples.

## Extremal epis as pairs of sublocales

As extremal epis in **Frm** can be represented uniquely by sublocale sets, we can represent an extremal epi in **dFrm**

$$h: L \rightarrow M$$

as a pair of sublocales  $S_- \in \mathcal{S}(L_-)$  and  $S_+ \in \mathcal{S}(L_+)$  endowed with  $\text{con}_S$  and  $\text{tot}_S$  relations induced by those of  $M$  (as  $S_- \simeq M_-$  and  $S_+ \simeq M_+$ ).

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*In that case,*

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- 2  $L_-^{\bullet\bullet} \subseteq S_-$  and  $L_+^{\bullet\bullet} \subseteq S_+$ .

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$$S = (S_-, S_+, \text{con}_L \cap S, \text{tot}_L \cap S)$$

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An extremal epi  $h: L \rightarrow S$  that is given by sublocales is dense iff  $L_-^{\bullet\bullet} \subseteq S_-$  and  $L_+^{\bullet\bullet} \subseteq S_+$ .

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## Theorem

Each  $d$ -frame has a least dense extremal epi.

Eskerrik asko arretarengatik!

Thank you for your attention!

Děkuji za pozornost!

`imanol.mozo@ehu.eus`

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