First-order logic properly displayed

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TACL 2017, Prague

Starting point: Display Calculi

- Natural generalization of Gentzen's sequent calculi;
- sequents $X \vdash Y$, where X and Y are **structures**:
  - formulas are **atomic structures**
  - built-up: **structural connectives** (generalizing meta-linguistic comma in sequents $\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$)
  - generation **trees** (generalizing sets, multisets, sequences)
- **Display property**:

$$\dfrac{\dfrac{\dfrac{Y \vdash X > Z}{X \,;\, Y \vdash Z}}{Y \,;\, X \vdash Z}}{X \vdash Y > Z}$$

display rules semantically justified by **adjunction/residuation**

- **Canonical proof of cut elimination (via metatheorem)**

Proper display calculi (Wansing 98)

Definition

A **proper display calculus** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation:
   - same shape, same position, non-proliferation;
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters (Properness!);
5. **reduction strategy** exists when both cut formulas are principal.

Theorem

Cut elimination and subformula property hold for any **proper display calculus**.

Multi-type proper display calculi

Definition

A **proper display calculus** verifies each of the following conditions:

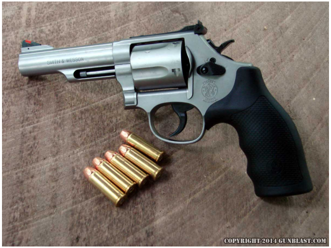
1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation (same shape, position, non-proliferation)
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters within each type (Properness!);
5. **reduction strategy** exists when cut formulas are principal.
6. **type-uniformity** of derivable sequents;
7. **strongly uniform cuts** in each/some type(s).

Theorem (Canonical!)

Cut elimination and subformula property hold for any **proper display calculus**.

Main Ideas

- Types: A finer book-keeping device for properness
- Display rules: Sliding doors between types
- 5 basic properties in a semi-automatic package



First-order logic and properness

$$\forall_L \; \frac{A[t/x], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \forall x A, \Delta} \; \forall_R$$

$$\exists_L \; \frac{A[y/x], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta} \; \exists_R$$

where in $\forall_R$ and $\exists_L$ y ***is not free in the conclusion***.

Display calculus: Quantifiers and adjunctions

Consider $\forall y : \wp(X \times Y) \to \wp(X)$ and $\pi^{-1} : \wp(X) \to \wp(X \times Y)$ defined as:

- $\forall y(A) = \bigcap_{y \in Y} \{x \in X \mid (x, y) \in A\}$
- $\pi^{-1}(A) = A \times Y$

We have:

$$\pi^{-1}(A) \subseteq B \quad \iff \quad A \subseteq \forall y(B)$$

Display calculus: Quantifiers and adjunctions

Consider $\exists y : \wp(X \times Y) \to \wp(X)$ and $\pi^{-1} : \wp(X) \to \wp(X \times Y)$ defined as:

- $\exists y(A) = \bigcup_{y \in Y}\{x \in X \mid (x, y) \in A\}$
- $\pi^{-1}(A) = A \times Y$

We have:

$$\exists y(A) \subseteq B \quad \Longleftrightarrow \quad A \subseteq \pi^{-1}(B)$$

Display calculus: Quantifiers and adjunctions

- **Algebraically:** Existential and universal quantification are the left and right adjoints respectively of the inverse projection map.
- **Categorically:** Existential and universal quantification are the left and right adjoints respectively of the pullback along projections.

Logical connectives and types

- Symbols for quantifiers and their adjoint for each $x \in Var$:

Structural symbols	$\mathtt{Q}_x$		$\circ_x$	
Operational symbols	$\exists x$	$\forall x$	$\cdot_x$	$\cdot_x$

- Types will be named after the elements $F \in \wp_\omega(Var)$.
- A type $\mathcal{L}_F$ contains a formula φ iff $\mathsf{FV}(\varphi) = F$.
- $\varphi \in \mathcal{L}_{F \cup \{y\}} \iff \forall y \varphi \in \mathcal{L}_F$
- $\psi \in \mathcal{L}_{F \setminus \{x\}} \iff \circ_x \psi \in \mathcal{L}_F$

Display Calculus

Introduction rules for quantifiers and their adjoint:

$$\exists_L \frac{\mathsf{Q}_x A \vdash_F X}{\exists x A \vdash_F X} \qquad \frac{X \vdash_F A}{\mathsf{Q} x X \vdash_{F\setminus\{x\}} \exists x A} \exists_R$$

$$\forall_L \frac{A \vdash_F X}{\forall x A \vdash_{F\setminus\{x\}} \mathsf{Q} x A} \qquad \frac{X \vdash_F \mathsf{Q}_x A}{X \vdash_F \forall x A} \forall_R$$

$$\circ_M \frac{X \vdash_{F\setminus\{x\}} Y}{\circ_x X \vdash_{F\cup\{x\}} \circ_x Y}$$

$$\cdot_L \frac{\circ_x A \vdash_F X}{\cdot x A \vdash_F X} \qquad \frac{X \vdash_F \circ_x A}{X \vdash_F \cdot x A} \cdot_R$$

Display Calculus

Display postulates for quantifiers and their adjoint:

$$\frac{\mathtt{Q}_x X \vdash_{F\setminus\{x\}} Y}{X \vdash_{F\cup\{x\}} \circ_x Y} \qquad \frac{Y \vdash_{F\setminus\{x\}} \mathtt{Q}_x X}{\circ_x Y \vdash_{F\cup\{x\}} X}$$

Necessitation quantification and their adjoint:

$$\frac{\mathrm{I} \vdash_F X}{\circ_x \mathrm{I} \vdash_{F\cup\{x\}} X} \qquad \frac{X \vdash_F \mathrm{I}}{X \vdash_{F\cup\{x\}} \circ_x \mathrm{I}}$$

Improper rules in light of multi-type

Assume that $x \notin \mathsf{FV}(Y)$. We have

$$\cfrac{\cfrac{A \vdash_F \circ_x Y}{\mathsf{Q}xA \vdash_{F\setminus\{x\}} Y}}{\exists xA \vdash_{F\setminus\{x\}} Y}$$

First-order logic and properness

$$\forall_L \frac{A[t/x], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \forall x A, \Delta} \forall_R$$

$$\exists_L \frac{A[y/x], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta} \exists_R$$

Variable substitution: Side conditions

- In $\forall x(Px \wedge Ry)$ the free variable y cannot be substituted with x.
- $\forall_L \dfrac{x = x \vdash x = x}{\forall y(y = x) \vdash x = x}$ is a valid proof.
- How to substitute x in the formula $\cdot_x A$?

Explicit substitution

- $(y//x)$: variable renaming;
- (t/x): substitution a term with fresh variables;
- (y/x): identifying two variables.

Substitution, as an explicit operation is both meet and join preserving, therefore it has both left and right adjoints.

Improper rules in light of substitution

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(t/x)A \vdash_F C}{A \vdash_{F'} [x/t]C}}{\forall x A \vdash_{F' \setminus \{x\}} \mathsf{Q}x[x/t]C}}{\circ_x \forall x A \vdash_{F'} [x/t]C}}{(t/x) \circ_x \forall x A \vdash_F C}}{\circ_{y_1} \ldots \circ_{y_m} \forall x A \vdash_{F' \setminus \{x\}} C}
$$

Expanded language $\mathcal{L}^\star$

- For every sequent in the language with explicit substitution, $\mathcal{L}^\star$, there exists a translation into a sequent in $\mathcal{L}$.
- For every provable sequent $X \vdash Y$ of the Gentzen calculus, there exists a provable sequent in $\mathcal{L}^\star$ whose translation is $X \vdash Y$.
- Given two sequents with the same translation, we cannot, in principle, show that one proves the other.

Interaction rules

$$\frac{(t/x)(X;Y) \vdash Z}{\overline{(t/x)X;(t/x)Y \vdash Z}} \qquad \frac{Z \vdash (t/x)(X;Y)}{\overline{Z \vdash (t/x)X;(t/x)Y}}$$

$$\frac{(t/x)\mathsf{Q}yX \vdash_{F\setminus\{z\}} Y}{\overline{\mathsf{Q}z(t/x)(z//y)X \vdash_{F\setminus\{z\}} Y}} \qquad \frac{Y \vdash_{F\setminus\{z\}} (t/x)\mathsf{Q}yX}{\overline{Y \vdash_{F\setminus\{z\}} \mathsf{Q}z(t/x)(z//y)X}}$$

$$\frac{(t/x)(s/y)X \vdash Y}{\overline{((t/x)s/y)X \vdash Y}} \qquad \frac{Y \vdash (t/x)(s/y)X}{\overline{Y \vdash ((t/x)s/y)X}}$$

if $x \in \mathsf{FV}(s)$.

New types

$$
\dfrac{
\dfrac{\dfrac{\dfrac{x \vdash x}{(y//x)x \vdash y} \qquad y \vdash y}{f(y,(y//x)x) \vdash f(y,y)}}{(y//x)f(y,x) \vdash f(y,y)}
\qquad
\dfrac{\dfrac{y \vdash y \qquad \dfrac{x \vdash x}{(y//x)x \vdash y}}{f(y,y) \vdash f((y//x)x,y)}}{f(y,y) \vdash (y//x)f(x,y)}
}{(y//x)f(y,x) \vdash (y//x)f(x,y)}
$$

Final message and questions

- Everything is explicit.
- Proper calculus.
- We can incorporate equational theories on the level of the types.
- More refined notions of quantification?
- Is adjunction meaningful on the level of the types?