First-order logic properly displayed

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Starting point: Display Calculi

- Natural generalization of Gentzen’s sequent calculi;
- sequents $X \vdash Y$, where $X$ and $Y$ are structures:
  - formulas are atomic structures
  - built-up: structural connectives (generalizing meta-linguistic comma in sequents $\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$)
  - generation trees (generalizing sets, multisets, sequences)

- Display property:

\[
\begin{align*}
Y \vdash X & > Z \\
X ; Y & \vdash Z \\
Y ; X & \vdash Z \\
X & \vdash Y > Z
\end{align*}
\]

display rules semantically justified by adjunction/residuation

- Canonical proof of cut elimination (via metatheorem)
Proper display calculi (Wansing 98)

Definition
A proper display calculus verifies each of the following conditions:

1. structures can disappear, formulas are forever;
2. tree-traceable formula-occurrences, via suitably defined congruence relation:
   - same shape, same position, non-proliferation;
3. principal = displayed
4. rules are closed under uniform substitution of congruent parameters (Properness!);
5. reduction strategy exists when both cut formulas are principal.

Theorem
Cut elimination and subformula property hold for any proper display calculus.
Multi-type proper display calculi

Definition
A proper display calculus verifies each of the following conditions:

1. structures can disappear, formulas are forever;
2. tree-traceable formula-occurrences, via suitably defined congruence relation (same shape, position, non-proliferation)
3. principal = displayed
4. rules are closed under uniform substitution of congruent parameters within each type (Properness!);
5. reduction strategy exists when cut formulas are principal.
6. type-uniformity of derivable sequents;
7. strongly uniform cuts in each/some type(s).

Theorem (Canonical!)
Cut elimination and subformula property hold for any proper display calculus.
Main Ideas

- Types: A finer book-keeping device for properness
- Display rules: Sliding doors between types
- 5 basic properties in a semi-automatic package
First-order logic and properness

\[\forall L \quad \frac{A[t/x], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta}\]
\[\forall R \quad \frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \forall x A, \Delta}\]
\[\exists L \quad \frac{A[y/x], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta}\]
\[\exists R \quad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta}\]

where in \(\forall R\) and \(\exists L\) \(y\) is not free in the conclusion.
Consider $\forall y : \wp(X \times Y) \rightarrow \wp(X)$ and $\pi^{-1} : \wp(X) \rightarrow \wp(X \times Y)$ defined as:

- $\forall y(A) = \bigcap_{y \in Y}\{x \in X \mid (x, y) \in A\}$
- $\pi^{-1}(A) = A \times Y$

We have:

$$\pi^{-1}(A) \subseteq B \iff A \subseteq \forall y(B)$$
Consider $\exists y : \wp(X \times Y) \rightarrow \wp(X)$ and $\pi^{-1} : \wp(X) \rightarrow \wp(X \times Y)$ defined as:

- $\exists y(A) = \bigcup_{y \in Y} \{x \in X \mid (x, y) \in A\}$
- $\pi^{-1}(A) = A \times Y$

We have:

$$\exists y(A) \subseteq B \iff A \subseteq \pi^{-1}(B)$$
Display calculus: Quantifiers and adjunctions

- **Algebraically:** Existential and universal quantification are the left and right adjoints respectively of the inverse projection map.

- **Categorically:** Existential and universal quantification are the left and right adjoints respectively of the pullback along projections.
Logical connectives and types

- Symbols for quantifiers and their adjoint for each \( x \in Var \):

<table>
<thead>
<tr>
<th>Structural symbols</th>
<th>( Q_x )</th>
<th>( \circ x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Operational symbols</td>
<td>( \exists x )</td>
<td>( \forall x )</td>
</tr>
</tbody>
</table>

- Types will be named after the elements \( F \in \wp(Var) \).
- A type \( \mathcal{L}_F \) contains a formula \( \varphi \) iff \( FV(\varphi) = F \).
- \( \varphi \in \mathcal{L}_{F \cup \{y\}} \iff \forall y \varphi \in \mathcal{L}_F \)
- \( \psi \in \mathcal{L}_{F \setminus \{x\}} \iff \circ x \psi \in \mathcal{L}_F \)
Introduction rules for quantifiers and their adjoint:

$\exists_L \quad \frac{\exists x A \vdash F X}{\exists x A \vdash F X}$ \quad $\exists_R \quad \frac{X \vdash F A}{Q_x X \vdash F \{x\} \exists x A}$

$\forall_L \quad \frac{A \vdash F X}{\forall x A \vdash F \{x\} Q x A}$ \quad $\forall_R \quad \frac{X \vdash F Q x A}{X \vdash F \forall x A}$

$\circ_M \quad \frac{X \vdash F \{x\} Y}{\circ_x X \vdash F \cup \{x\} \circ_x Y}$

$\cdot_L \quad \frac{\circ_x A \vdash F X}{\cdot x A \vdash F X}$ \quad $\cdot_R \quad \frac{X \vdash F \circ x A}{X \vdash F \cdot x A}$
Display Calculus

Display postulates for quantifiers and their adjoint:

\[
\begin{align*}
Q_x X & \vdash F\{x\} Y \\
X & \vdash F \cup \{x\} \circ_x Y \\
Y & \vdash F\{x\} Q_x X \\
\circ_x Y & \vdash F \cup \{x\} X
\end{align*}
\]

Necessitation quantification and their adjoint:

\[
\begin{align*}
I & \vdash F X \\
\circ_x I & \vdash F \cup \{x\} X \\
X & \vdash F I \\
X & \vdash F \cup \{x\} \circ_x I
\end{align*}
\]
Assume that $x \not\in \text{FV}(Y)$. We have

$$A \vdash F \circ x Y$$

$$QxA \vdash F\{x\}Y$$

$$\exists x A \vdash F\{x\}Y$$
First-order logic and properness

\[ \forall_L \quad \frac{A[t/x], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \quad \forall_R \quad \frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \forall x A, \Delta} \]

\[ \exists_L \quad \frac{A[y/x], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta} \quad \exists_R \quad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta} \]
Variable substitution: Side conditions

▶ In $\forall x(Px \land Ry)$ the free variable $y$ cannot be substituted with $x$.

▶ $\forall L \quad \frac{x = x \vdash x = x}{\forall y(y = x) \vdash x = x}$ is a valid proof.

▶ How to substitute $x$ in the formula $\cdot_x A$?
Explicit substitution

- $(y/x)$: variable renaming;
- $(t/x)$: substitution a term with fresh variables;
- $(y/x)$: identifying two variables.

Substitution, as an explicit operation is both meet and join preserving, therefore it has both left and right adjoints.
Improper rules in light of substitution

\[
\begin{align*}
(t/x)A &\vdash FC \\
A &\vdash F'[x/t]C \\
\forall xA &\vdash F'[\{x\} \backslash Q x[x/t]C \\
(\circ_x \forall xA) &\vdash F'[x/t]C \\
(t/x) \circ_x \forall xA &\vdash FC \\
(\circ_{y_1} \ldots \circ_{y_m} \forall xA) &\vdash F'[\{x\} \backslash C
\end{align*}
\]
For every sequent in the language with explicit substitution, $\mathcal{L}^*$, there exists a translation into a sequent in $\mathcal{L}$.

For every provable sequent $X \vdash Y$ of the Gentzen calculus, there exists a provable sequent in $\mathcal{L}^*$ whose translation is $X \vdash Y$.

Given two sequents with the same translation, we cannot, in principle, show that one proves the other.
Interaction rules

\[
\begin{align*}
(t/x)(X; Y) & \vdash Z \\
(t/x)X; (t/x)Y & \vdash Z \\
(t/x)QyX & \vdash F\{z\} Y \\
Qz(t/x)(z/y)X & \vdash F\{z\} Y \\
(t/x)(s/y)X & \vdash Y \\
((t/x)s/y)X & \vdash Y \\
Y & \vdash (t/x)(s/y)X \\
Y & \vdash ((t/x)s/y)X
\end{align*}
\]

if \( x \in \text{FV}(s) \).
New types

\[
\begin{align*}
& x \vdash x \\
& \frac{(y\mathbin{//}x)x \vdash y}{f(y, (y\mathbin{//}x)x) \vdash f(y, y)} \quad \frac{y \vdash y}{f(y, y) \vdash f((y\mathbin{//}x)x, y)} \\
& \frac{(y\mathbin{//}x)f(y, x) \vdash f(y, y)}{(y\mathbin{//}x)f(y, x) \vdash (y\mathbin{//}x)f(x, y)}
\end{align*}
\]
Final message and questions

- Everything is explicit.
- Proper calculus.
- We can incorporate equational theories on the level of the types.
- More refined notions of quantification?
- Is adjunction meaningful on the level of the types?