Fixed-point elimination in Heyting algebras[1]

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[1] See [Ghilardi et al., 2016]

Plan

μ-calculi

Add to a given algebraic framework
syntactic least and greatest fixed-point constructors.

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E.g., the propositional modal μ-calculus:

$$\phi := x \mid \neg x \mid \top \mid \phi \wedge \phi \mid \bot \mid \phi \vee \phi \mid \Box\phi \mid \Diamond\phi$$
$$\mid \mu_x.\phi \mid \nu_x.\phi\,, \qquad \text{when } x \text{ is positive in } \phi.$$

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Interpret the syntactic least (resp. greatest) fixed-point as expected.

$[\![\mu_x.\phi]\!]_v :=$

least fixed-point of the monotone mapping $X \mapsto [\![\phi]\!]_{v,X/x}$

Alternation hierarchies in μ-calculi

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Problem. For a given μ-calculus, does there exist n such that, for each ϕ with $\sharp\phi > n$, there exists ψ with $\gamma \equiv \psi$ and $\sharp\psi \leq n$?

Alternation hierarchies, facts

- The alternation hierarchy for the modal μ-calculus is infinite (there exists no such n) [Lenzi, 1996, Bradfield, 1998].
- Idem for the lattice μ-calculus [Santocanale, 2002].
- The alternation hierarchy for the linear μ-calculus ($\diamond x = \Box x$) is reduced to the Büchi fragment (here $n = 2$) .
- The alternation hierarchy for the modal μ-calculus on transitive frames collapses to the alternation free fragment (here $n = 1.5$) [Alberucci and Facchini, 2009].
- The alternation hierarchy for the distributive μ-calculus is trivial (here $n = 0$) [Kozen, 1983].

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A research plan:

Develop a theory explaining why alternation hierarchies collapses.

μ-calculi on generalized distributive lattices

Theorem. [Frittella and Santocanale, 2014] There are lattice varieties (Nation's varieties)

$$\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \ldots \subseteq \mathcal{D}_n \subseteq \ldots$$

with $\mathcal{D}_0$ the variety of distributive lattices, such that, on $\mathcal{D}_n$ and for any lattice term ϕ,

$$\phi^{n+2}(\perp) = \phi^{n+1}(\perp) \ \ (= \mu_x.\phi)\,, \qquad \phi^{n}(\perp) \neq \phi^{n+1}(\perp)\,.$$

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Corollary. The alternation hierarchy of the lattice μ-calculus is trivial on $\mathcal{D}_n$, for each $n \geq 0$.

Plan

A primer on mu-calculi

The intuitionistic μ-calculus

The elimination procedure

Bounding closure ordinals

The intuitionistic μ-calculus

After the distributive μ-calculus, the next on the list—by Pitt's quantifiers, we knew that least fixed-points and greatest fixed-points are definable.

We extend the signature of Heyting algebras (i.e. Intuitionistic Logic) with least and greatest fixed-point constructors.

Intuitionistic μ-terms are generated by the grammar:

$$\boxed{\begin{aligned} \phi := x \mid \top \mid \phi \wedge \phi \mid \bot \mid \phi \vee \phi \mid \phi \to \phi \\ \mid \mu_x.\phi \mid \nu_x.\phi\,, \qquad \text{when } x \text{ is positive in } \phi. \end{aligned}}$$

Heyting algebra semantics

We take any *provability* semantics of IL with fixed points:

- (Complete) Heyting algebras.
- Kripke frames.
- Any sequent calculus for Intuitionisitc Logic (e.g. LJ) plus Park/Kozen's rules for least and greatest fixed-points:

$$\frac{\phi[\psi/x] \vdash \psi}{\mu_x.\phi \dashv \psi} \qquad \frac{\Gamma \vdash \phi(\mu_x.\phi)}{\Gamma \vdash \mu_x.\phi} \quad \frac{\phi(\nu_x.\phi) \vdash \delta}{\nu_x.\phi \vdash \delta} \qquad \frac{\psi \vdash \phi[\psi/x]}{\psi \vdash \nu_x.\phi}$$

Definition. A *Heyting algebra* is a bounded lattice $H = \langle H, \top, \wedge, \bot, \vee \rangle$ with an additional binary operation $\rightarrow$ satisfying

$$x \wedge y \leq z \quad \text{iff} \quad x \leq y \rightarrow z\,.$$

Ruitenburg's theorem [Ruitenburg, 1984]

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$$\phi^n(\bot) \leq \phi^{n+1}(\bot) \leq \phi^{n+2}(\bot) = \phi^n(\bot)\,,$$

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Corollary. The alternation hierarchy for the intuitionistic μ-calculus is trivial.

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NB : Ruitenburg's n might not be the *closure ordinal* of $\mu_x.\phi$.

Peirce, compatibility, strenghs and strongness

Proposition. Peirce's theorem for Heyting algebras.
Every term ϕ is compatible. In particular, for ψ, χ arbitrary terms, the equation

$$\phi[\psi/x] \wedge \chi = \phi[\psi \wedge \chi/x] \wedge \chi\,.$$

holds on Heyting algebras.

Corollary. Every term ϕ monotone in x is *strong* in x. That is, any the following equivalent conditions

$$\phi[\psi/x] \wedge \chi \leq \phi[\psi \wedge \chi/x]\,, \qquad \psi \to \chi \leq \phi[\psi/x] \to \phi[\chi/x],$$

hold, for any terms ψ and χ.

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Using strongness:

$$\phi(\top) = \phi(\top) \wedge \phi(\top) \leq \phi(\top \wedge \phi(\top)) = \phi^2(\top)\,.$$

Greatest solutions of systems of equations

Proposition. On Heyting algebras, a system of equations

$$\left\{ \begin{array}{ll} x_1 & = \phi_1(x_1, \ldots, x_n) \\ \vdots & \\ x_n & = \phi_n(x_1, \ldots, x_n) \end{array} \right\}$$

has a greatest solution obtained by iterating

$$\phi := \langle \phi_1, \ldots, \phi_n \rangle$$

n times from $\top$.

Proof. Using the Bekic property.

Least fixed-points: splitting the roles of variables

Due to

$$\boxed{\mu_x.\phi(x,x) = \mu_x.\mu_y.\phi(x,y)}$$

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we can separate computing the least fixed-points w.r.t:

weakly negative variables: variables that appear within the left-hand-side of an implication,

fully positive variables: those appearing only within the right-hand-side of an implication.

Weakly negative least fixed-points: an example

Use

$$\boxed{\mu_x.(f \circ g)(x) = f(\, \mu_y.(g \circ f)(y)\,)}$$

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to argue that:

$$\begin{aligned}
\mu_x.[\,(x \to a) \to b\,] &= [\,\nu_y.(y \to b) \to a\,] \to b \\
&= [\,(\top \to b) \to a\,] \to b \\
&= [\,b \to a\,] \to b\,.
\end{aligned}$$

Weakly negative least fixed-points: reducing to greatest fixed-points

If each occurrence of x in ϕ is weakly negative, then

$$\phi(x) = \phi_0[\phi_1(x)/y_1, \ldots, \phi_n(x)/y_n]$$

with $\phi_0(y_1, \ldots, y_n)$ negative in each y_j.

Due to

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Due to

$$\boxed{\mu_x.(f \circ g)(x) = f(\,\mu_y.(g \circ f)(y)\,)}$$

we have

$$\boxed{\begin{aligned} \mu_x.\phi(x) &= \mu_x.(\,\phi_0 \circ \langle \phi_1, \ldots, \phi_n \rangle\,)(x) \\ &= \phi_0(\,\nu_{y_1 \ldots y_n}.(\,\langle \phi_1, \ldots \phi_n \rangle \circ \phi_0\,)(y_1, \ldots y_n))\,. \end{aligned}}$$

Interlude: least fixed-points of strong functions

If f and f_i, $i \in I$, are strong, then

$$\mu_x.a \wedge f(x) = a \wedge \mu_x.f(x)\,,$$
$$\mu_x.\bigwedge_{i \in I} f_i(x) = \bigwedge \mu_x.f_i(x)\,,$$
$$\mu_x.a \to f(x) = a \to \mu_x.f(x)\,.$$

Strongly positive fixed-points: disjunctive formulas

The equation

$$\mu_x . \bigwedge_{i \in I} f_i(x) = \bigwedge_{i \in I} \mu_x . f_i(x)$$

allows to push least fixed-points down through conjunctions.

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Once all conjunctions have been pushed up in formulas, we are left to compute least fixed-points of *disjunctive formulas*, generated by the grammar:

$$\phi = x \mid \beta \vee \phi \mid \alpha \to \phi \mid \bigvee_{i=1,\ldots,n} \phi_i \,,$$

where α and β do not contain the variable x.

We call α an *head subformula* and β a *side subformula*.

Least fixed-points of inflating functions

All functions f denoted by such formula ϕ are (monotone and) *inflating*:

$$x \leq f(x)\,.$$

Let f_i, $i = 1, \ldots, n$, be a collection of monotone inflating functions. Then

$$\boxed{\mu_x. \bigvee_{i=1,\ldots,n} f_i(x) = \mu_x.(f_1 \circ \ldots \circ f_n)(x)\,.}$$

Least fixed-points of disjunctive formulas

Proposition. Let ϕ be a disjunctive formula, with $\mathit{Head}(\phi)$ (resp., $\mathit{Side}(\phi)$) the collection of its head (resp., side) subformulas. Then

$$\mu_x.\phi = \bigwedge_{\alpha \in \mathit{Head}(\phi)} \alpha \to \bigvee_{\beta \in \mathit{Side}(\phi)} \beta \,.$$

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If $\mathit{Head}(\phi) = \{\, \alpha_1, \ldots, \alpha_n \,\}$ and $\mathit{Side}(\phi) = \{\, \beta_1, \ldots, \beta_m \,\}$:

$$\begin{aligned}
\mu_x.\phi &= \mu_x.\alpha_1 \to \alpha_2 \to \ldots \to \alpha_n \to \beta_1 \vee \ldots \vee \beta_m \vee x \\
&= \mu_x. \bigwedge_{\alpha \in \mathit{Head}(\phi)} \alpha \to \bigvee_{\beta \in \mathit{Side}(\phi)} \beta \vee x \\
&= \bigwedge_{\alpha \in \mathit{Head}(\phi)} \alpha \to \mu_x. \bigvee_{\beta \in \mathit{Side}(\phi)} \beta \vee x \\
&= \bigwedge_{\alpha \in \mathit{Head}(\phi)} \alpha \to \bigvee_{\beta \in \mathit{Side}(\phi)} \beta \,.
\end{aligned}$$

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Closure ordinals

Definition. (*Closure ordinal*). For $\mathcal{K}$ a class of models and $\phi(x)$ a monotone formula/term, let

$$\mathtt{cl}_{\mathcal{K}}(\phi) = \text{least ordinal } \alpha \text{ such that } \mathcal{M} \models \mu_x.\phi = \phi^{\alpha}(\bot)\,.$$

In general, $\mathtt{cl}_{\mathcal{K}}(\phi)$ might not exist.

If $\mathcal{H}$ is the class of Heyting algebras and $\phi(x)$ is an intuitionistic formula, then

$$\mathtt{cl}_{\mathcal{H}}(\phi) < \omega\,.$$

Upper bounds from fixed-point equations

$$\mathtt{cl}_{\mathcal{K}}(\, f \circ g \,) \leq \mathtt{cl}_{\mathcal{K}}(\, g \circ f \,) + 1,$$

$$\mathtt{cl}_{\mathcal{H}}(\, \phi_0(\phi_1(x), \ldots, \phi_n(x)) \,) \leq n + 1,$$

when $\phi_0(y_1, \ldots, y_n)$ contravariant,

$$\mathtt{cl}_{\mathcal{H}}(\, \phi \,) \leq \mathbf{card}(\mathit{Head}(\phi)) + 1\,,$$

when ϕ is a disjunctive formula,

$$\mathtt{cl}_{\mathcal{K}}(\, f \circ \Delta \,) \leq n \cdot \mathtt{cl}_{\mathcal{K}}(g),$$

when $n = \mathtt{cl}_{\mathcal{K}}(f(x, \_))$ and $g(x) = \mu_y.f(x, y)$,

$$\mathtt{cl}_{\mathcal{K}}(f \wedge g) \leq \mathtt{cl}_{\mathcal{K}}(f) + \mathtt{cl}_{\mathcal{H}}(g) - 1,$$

when f and g are strong.

Examples

1. Weakly negative x:

$$\bigwedge_{i=1,\ldots,n} (x \to a_i) \to b_i$$

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2. Fully positive x:

$$b \vee \bigvee_{i=1,\ldots,n} a_i \to x$$

converges after $n+1$ steps. This upper bound is strict.

Examples (II)

- Similarly,

$$\phi(x) := \bigvee_{i=1,\ldots,n} (x \to a_i) \to b_i$$

converges within $n+1$ steps, according to the general theory.

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Theorem. For any $n \geq 2$, $\phi(x)$ converges to its least fixed-point within 3 steps.

Back to Ruitenburg's theorem

- Inspection of Ruitenburg's paper shows that

$$\mathtt{cl}_{\mathcal{H}}(\phi) = O(n)\,,$$

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- Given a fully positive formula ϕ, pushing up conjuctions yields a formula

$$\bigwedge_{i=1,\ldots,k} \delta_i\,, \quad \delta_i \text{ disjunctive,}$$

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Our method yields the upper bound

$$\mathtt{cl}_{\mathcal{H}}(\phi) = \mathtt{cl}_{\mathcal{H}}(\bigwedge_{i=1,\ldots,k} \delta_i) \leq 1 - k + \sum_{i=1,\ldots,k} \mathtt{cl}_{\mathcal{H}}(\delta_i)\,,$$

exponential w.r.t. the size of ϕ.

Closing the gap ?

Problem. Let $\delta_1, \delta_2, \ldots, \delta_n$ be disjunctive formulas and put

$$\phi(x) := \bigwedge_{i=1,\ldots,n} \delta_i(x)\,.$$

Does

$$\mu_x.\phi(x) = \bigwedge_{i=1,\ldots,n} \bigwedge Head(\delta_i) \to \bigvee Side(\delta_i) \;\leq \phi^{H+1}(\bot)\,,$$

with $H = \mathbf{card}(\bigcup_{i=1,\ldots,n} Head(\delta_i))$?

- This holds (non trivially) for $n = 2$.

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- This holds (non trivially) for $n = 2$.
- Not the only plausible conjecture.

After thoughts

- A decision procedure for the Intuitionistic μ-calculus.
- Axiomatization of fixed-points and of some Pitt's quantifiers.
- General theory of fixed-point elimination: no uniform upper bounds for closure ordinals.
- Relevance of strongness
  — it looks like Pitt's quantifiers less relevant.
- A working path to understand Ruitenburg's theorem.

Thanks ! Questions ?

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