Filter pairs: A new way of presenting logics

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A logic is a pair (Σ,\vdash) with

- Σ a signature of finitary connectives
- $-\vdash \subseteq \mathcal{P}(Fm_{\Sigma}(X)) \times Fm_{\Sigma}(X)$ monotonous, increasing, idempotent, substitution invariant and finitary

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Finitarity: The lattice is an *algebraic lattice*. Every element is the supremum of the finitely presented elements

Structurality:

$$\begin{array}{c|c} Fm_{\Sigma}(X) & Th \stackrel{i}{\longrightarrow} \mathcal{P}(Fm_{\Sigma}(X)) \\ \sigma & & \sigma^{-1}|_{Th} & \uparrow \sigma^{-1} \\ Fm_{\Sigma}(X) & Th \stackrel{i}{\longmapsto} \mathcal{P}(Fm_{\Sigma}(X)) \end{array}$$

The inclusion of the theories is a natural transformation.

Definition: Let Σ be a signature.

(i) A (finitary) *filter pair* is a pair (G, i) consisting of

– a functor $G: \Sigma$ - $Alg^{op} \rightarrow AlgLat$

– a natural transformation $i: G \to \mathcal{P}(-)$ which, at each object, preserves arbitrary infima and directed suprema.

$$\begin{array}{ccc} A & & G(A) \xrightarrow{i} \mathcal{P}(A) \\ h & & h^* & \uparrow & \uparrow h^{-1} \\ B & & G(B) \xrightarrow{i} \mathcal{P}(B) \end{array}$$

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(ii) The logic associated to a filter pair (G, i) is the logic associated to the algebraic lattice given by the image $i(G(Fm_{\Sigma}(X))) \subseteq \wp(Fm_{\Sigma}(X))$.

(i) The logic associated to a filter pair is finitary and structural.
(ii) If L is a logic, then there is a filter pair (Fi_L, i), given by

$$\Sigma$$
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Note:

- preimages of *L*-filters under algebra homomorphisms are *L*-filters. Hence Fi_L is a well-defined functor and *i* a natural transformation.
- every logic comes from a filter pair.
- the associated logic only depends on the image of $i_{Fm_{\Sigma}(X)}$.
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- every logic comes from a filter pair.
- the associated logic only depends on the image of $i_{Fm_{\Sigma}(X)}$.
- a filter pair can be seen as a *presentation* of a logic.
- i_A always maps into filters of the associated logic. But for $A \neq Fm_{\Sigma}(X)$ not every filter needs to arise in this way.

Let K be a quasivariety. Then

$$Co_{\mathcal{K}}(\mathcal{A}) := \{ \text{congruences } \theta \text{ on A s.t. } \mathcal{A}/\theta \in \mathcal{K} \}$$

is a functor Σ -Str^{op} \rightarrow AlgLat.

A filter pair of the form (Co_K, i) is called *congruence filter pair*.

Theorem:

The logic presented by a congruence filter pair (Co_K, i) with *i* injective is algebraizable with associated quasivariety *K*.

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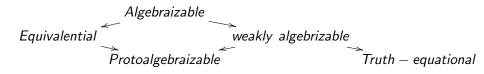
Let K be a quasivariety, and $au=\langle\epsilon,\delta
angle$ a set of equations Then

 $(G: A \mapsto Con_{\mathcal{K}}(A), i: \theta \mapsto \{a \in A \mid \epsilon(a) = \delta(a) \text{ in } A/\theta\})$

defines a filter pair. Such a filter pair is called equational filter pair.

Example:

Let L be an algebraizable logic with associated quasivariety K. Then by Blok-Pigozzi there exists a set of equations τ inducing the isomorphism between congruences and theories



Logics coming from equational filter pairs need not be protoalgebraic, truth-equational or self-extensional.

Example: $\Sigma = \{ \lor, \land, \neg, \top, \bot \}$, *K*=variety of pseudocomplemented lattices, $\tau = \langle, x, \top \rangle$ yields **IPC**^{*}, the implicationless fragment of intuitionistic propositional calculus. By [Blok-Pigozzi, Algebraizable logics, Thm. 5.13] this is not protoalgebraic.

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Example: $\Sigma = \{s^{[1]}\}, K = \Sigma$ -*Str*, $\tau = \langle x, s(x) \rangle$. The absolutely free algebra $Fm_{\Sigma}(X)$ consists of countably many copies of the natural numbers with *s* the successor operation. Thus there are no *s*-fixed points in $Fm_{\Sigma}(X)/\theta_{min} = Fm_{\Sigma}(X)$, i.e. the set of theorems $i(\theta_{min}) = \{\varphi \in Fm_{\Sigma}(X) \mid \varphi = s(\varphi)\}$ is empty. Hence the logic is neither protoalgebraic nor truth-equational.

Proposition: Let $(Co_{\mathcal{K}}, i)$ be an equational filter pair given by equations $\tau = \langle \delta, \epsilon \rangle$, and let L be the associated logic.

- (i) *L* has an algebraic semantics in *K*, i.e. $\Gamma \vdash_L \varphi \Leftrightarrow \{\delta(\gamma) = \epsilon(\gamma) \mid \gamma \in \Gamma\} \vDash_K \delta(\varphi) = \epsilon(\varphi)$ (ii) If the set of th
- (ii) If *i* is injective, then *L* is algebraizable with τ being one half of an algebraizing pair.
- (iii) If *i* is surjective *onto filters*, then *L* is truth-equational.

Let Σ be a signature and $K \subseteq \Sigma - Str$ a class of algebras.

Theorem:

Let (Co_K, i) be an equational filter pair and L the associated logic. If K has the matrix-amalgamation property restricted to reduced matrices, then L has the Craig interpolation property.

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Let (Co_K, i) be an equational filter pair and assume the associated logic L is truth-equational. If K has the amalgamation property, then L has the Craig interpolation property.

(Example: if the equation is of the form $\langle x, \top \rangle$, the logic is always truth-equational.)

Suppose $\Sigma \supseteq \{\land, \lor, \top, \bot, \neg\}$, *K* is a quasivariety of lattices and we are interested in the assertional logic *L* given by $\langle x, \top \rangle$.

We can study the logics associated to new equations, e.g. $\langle x, \neg x \rangle$, $\langle x \land \neg x, \bot \rangle$, $\langle x \lor \neg x, \top \rangle$ or $\langle x, \neg \neg x \rangle$.

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Theorem: Suppose we know that L is equivalential, has the deduction detachment peoperty and satisfies Craig interpolation. Then K has the amalgamation property.

Now to establish Craig interpolation for the new logics, it suffices to check matrix amalgamation for the new reduced matrices...

Adjoint functor theorem for posets:

Let $L: X \to Y$ be an order preserving map of posets, X having arbitrary joins.

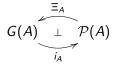
Then L has a right adjoint $R: Y \to X$ iff it preserves all joins. It is then given by $R(y) := \bigvee \{x \mid L(x) \le y\}.$

Dually a function from a poset with all meets has a left adjoint iff it preserves all meets. It is then given by $L(x) := \bigwedge \{y \mid x \le R(y)\}.$

(see e.g. Taylor, Practical foundations for mathematics, Thm.3.6.9)

Corollary:

Let (G, i) be a filter pair. Then i has a left adjoint Ξ .



The closure operator of the associated logic *L* is $i_{Fm_{\Sigma}(X)} \circ \Xi_{Fm_{\Sigma}(X)}$.

More generally, for every algebra A we get two g-matrices, $\langle A, Fi_L(A) \rangle$ and $\langle A, \{(i_A \circ \Xi_A)\text{-closed sets}\}\rangle$.

Consider an equational filter pair $(Co_{\mathcal{K}}, i)$: The left adjoint to $i: Co_{\mathcal{K}} \to \wp(A), \ \theta \mapsto \{a \mid \langle \delta(a), \epsilon(a) \rangle \in \theta\}$ is given by

 $\Xi(F) = [\text{the } K\text{-congruence generated by } \{ \langle \delta(a), \epsilon(a) \rangle \mid a \in F \}]$

The associated logic L is protoalgebraic iff Ω is monotonous iff Ω preserves arbitrary infima iff Ω has a left adjoint ℓ

$$\ell(\theta) = \bigwedge \{F \mid \theta \subseteq \Omega(F)\} \\ = \bigcap \{F \mid F \text{ is union of } \theta \text{-equivalence classes}\} \\ = \bigcup \{\theta \text{-equivalence classes of theorems of } I\} \\ = \bigcup \{[\varphi]_{\theta} \mid \emptyset \vdash_{I} \varphi\}$$

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(Compare: If the equation is given by $\langle, \top \rangle$, then $i(\theta) = [\top]_{\theta}$)

Let $(Co_{\mathcal{K}}, i)$ be an equational filter pair. Let A be a Σ -algebra and $\theta \in Co_{\mathcal{K}}(A)$ a congruence on A. Then $\Xi_A(i(\theta)) \subseteq \theta \subseteq \Omega_A(i_A(\theta))$

Let $(Co_{\mathcal{K}}, i)$ be an equational filter pair. Let A be a Σ -algebra and $\theta \in Co_{\mathcal{K}}(A)$ a congruence on A. Then $\Xi_A(i(\theta)) \subseteq \theta \subseteq \Omega_A(i_A(\theta))$

Proposition: The following are equivalent:

(i) Both of the inclusions are equalities

(ii) One of the inclusions is an equality

(iii) *i* is injective

If these conditions hold, then:

• L is algebraizable, with associated quasivariety K

•
$$\Omega = \Xi = i^{-1}$$

- A. Further bridge theorems like Craig interpolation.
- B. Infinitary version (work in progress)
- C. Further examples of filter pairs (e.g. relation to non-deterministic semantics)