Filter pairs: A new way of presenting logics

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Logics

A *logic* is a pair $(\Sigma, \vdash)$ with

- Σ a signature of finitary connectives
- $\vdash \subseteq \mathcal{P}(Fm_\Sigma(X)) \times Fm_\Sigma(X)$ monotonous, increasing, idempotent, substitution invariant and finitary

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Finitarity: The lattice is an *algebraic lattice*. Every element is the supremum of the finitely presented elements

Structurality:

$$\begin{tikzcd} Fm_\Sigma(X) \arrow[d, "\sigma"'] & Th \arrow[r, hook, "i"] & \mathcal{P}(Fm_\Sigma(X)) \\ Fm_\Sigma(X) & Th \arrow[u, "\sigma^{-1}|_{Th}"] \arrow[r, hook, "i"] & \mathcal{P}(Fm_\Sigma(X)) \arrow[u, "\sigma^{-1}"'] \end{tikzcd}$$

The inclusion of the theories is a *natural transformation*.

Finitary filter pairs

Definition: Let Σ be a signature.

(i) A (finitary) *filter pair* is a pair (G, i) consisting of

– a functor $G\colon \Sigma\text{-}\mathit{Alg}^{op} \to \mathit{AlgLat}$

– a natural transformation $i\colon G \to \mathcal{P}(-)$ which, at each object, preserves arbitrary infima and directed suprema.

$$\begin{tikzcd} A \arrow[d, "h"'] & G(A) \arrow[r, "i"] & \mathcal{P}(A) \\ B & G(B) \arrow[u, "h^*"] \arrow[r, "i"] & \mathcal{P}(B) \arrow[u, "h^{-1}"'] \end{tikzcd}$$

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$$
\begin{array}{ccc}
A & & \begin{array}{ccc} G(A) & \xrightarrow{i} & \mathcal{P}(A) \\ {\scriptstyle h^*}\uparrow & & \uparrow{\scriptstyle h^{-1}} \\ G(B) & \xrightarrow{i} & \mathcal{P}(B) \end{array} \\
{\scriptstyle h}\downarrow & & \\
B & &
\end{array}
$$

(ii) The *logic associated to a filter pair* (G, i) is the logic associated to the algebraic lattice given by the image $i(G(Fm_\Sigma(X))) \subseteq \wp(Fm_\Sigma(X))$.

Proposition:

(i) The logic associated to a filter pair is finitary and structural.

(ii) If L is a logic, then there is a filter pair (Fi_L, i), given by

$$\Sigma\text{-}Alg \ni A \quad \mapsto \quad (i_A \colon Fi_L(A) := \{L\text{-filters in } A\} \hookrightarrow \mathcal{P}(A))$$

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Note:

– preimages of L-filters under algebra homomorphisms are L-filters. Hence Fi_L is a well-defined functor and i a natural transformation.

– every logic comes from a filter pair.

– the associated logic only depends on the image of $i_{Fm_\Sigma(X)}$.

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- the associated logic only depends on the image of $i_{Fm_\Sigma(X)}$.
- a filter pair can be seen as a *presentation* of a logic.
- i_A always maps into filters of the associated logic. But for $A \neq Fm_\Sigma(X)$ not every filter needs to arise in this way.

Congruence filter pairs

Let K be a quasivariety. Then

$$Co_K(A) := \{\text{congruences } \theta \text{ on A s.t. } A/\theta \in K\}$$

is a functor $\Sigma\text{-}Str^{op} \to AlgLat$.

A filter pair of the form (Co_K, i) is called *congruence filter pair*.

Theorem:
The logic presented by a congruence filter pair (Co_K, i) with i injective is algebraizable with associated quasivariety K.

Equational filter pairs

Theorem:
Let K be a quasivariety, and $\tau = \langle \epsilon, \delta \rangle$ a set of equations Then

$$(G\colon A \mapsto Con_K(A), \quad i\colon \theta \mapsto \{a \in A \mid \epsilon(a) = \delta(a) \text{ in } A/\theta\})$$

defines a filter pair. Such a filter pair is called *equational filter pair*.

Example:
Let L be an algebraizable logic with associated quasivariety K. Then by Blok-Pigozzi there exists a set of equations τ inducing the isomorphism between congruences and theories

A bit of the Leibniz hierarchy

$$\begin{tikzcd}
 & Algebraizable \arrow[dl] \arrow[dr] & & \\
Equivalential \arrow[dr] & & weakly\ \ algebrizable \arrow[dl] \arrow[dr] & \\
 & Protoalgebraizable & & Truth-equational
\end{tikzcd}$$

Logics from equational filter pairs

Logics coming from equational filter pairs need not be protoalgebraic, truth-equational or self-extensional.

Example: $\Sigma = \{\vee, \wedge, \neg, \top, \bot\}$, K=variety of pseudocomplemented lattices, $\tau = \langle, x, \top\rangle$ yields $\mathbf{IPC}^*$, the implicationless fragment of intuitionistic propositional calculus. By [Blok-Pigozzi, Algebraizable logics, Thm. 5.13] this is not protoalgebraic.

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Example: $\Sigma = \{s^{[1]}\}$, $K = \Sigma\text{-}Str$, $\tau = \langle x, s(x) \rangle$. The absolutely free algebra $Fm_\Sigma(X)$ consists of countably many copies of the natural numbers with s the successor operation. Thus there are no s-fixed points in $Fm_\Sigma(X)/\theta_{min} = Fm_\Sigma(X)$, i.e. the set of theorems $i(\theta_{min}) = \{\varphi \in Fm_\Sigma(X) \mid \varphi = s(\varphi)\}$ is empty. Hence the logic is neither protoalgebraic nor truth-equational.

Logics from equational filter pairs

Proposition: Let (Co_K, i) be an equational filter pair given by equations $\tau = \langle \delta, \epsilon \rangle$, and let L be the associated logic.

(i) L has an algebraic semantics in K, i.e.
$\Gamma \vdash_L \varphi \Leftrightarrow \{\delta(\gamma) = \epsilon(\gamma) \mid \gamma \in \Gamma\} \vDash_K \delta(\varphi) = \epsilon(\varphi)$

(ii) If i is injective, then L is algebraizable with τ being one half of an algebraizing pair.

(iii) If i is surjective *onto filters*, then L is truth-equational.

Craig interpolation

Let Σ be a signature and $K \subseteq \Sigma - Str$ a class of algebras.

Theorem:
Let (Co_K, i) be an equational filter pair and L the associated logic. If K has the matrix-amalgamation property restricted to reduced matrices, then L has the Craig interpolation property.

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Theorem:
Let (Co_K, i) be an equational filter pair and assume the associated logic L is truth-equational. If K has the amalgamation property, then L has the Craig interpolation property.

(Example: if the equation is of the form $\langle x, \top \rangle$, the logic is always truth-equational.)

A pattern of application: Varying equations

Suppose $\Sigma \supseteq \{\wedge, \vee, \top, \bot, \neg\}$, K is a quasivariety of lattices and we are interested in the assertional logic L given by $\langle x, \top \rangle$.

We can study the logics associated to new equations, e.g. $\langle x, \neg x \rangle$, $\langle x \wedge \neg x, \bot \rangle$, $\langle x \vee \neg x, \top \rangle$ or $\langle x, \neg\neg x \rangle$.

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Theorem: Suppose we know that L is equivalential, has the deduction detachment peoperty and satisfies Craig interpolation. Then K has the amalgamation property.

Now to establish Craig interpolation for the new logics, it suffices to check matrix amalgamation for the new reduced matrices...

Reminder on Galois connections

Adjoint functor theorem for posets:
Let $L\colon X \to Y$ be an order preserving map of posets, X having arbitrary joins.
Then L has a right adjoint $R\colon Y \to X$ iff it preserves all joins. It is then given by $R(y) := \bigvee\{x \mid L(x) \leq y\}$.
Dually a function from a poset with all meets has a left adjoint iff it preserves all meets. It is then given by $L(x) := \bigwedge\{y \mid x \leq R(y)\}$.

(see e.g. Taylor, Practical foundations for mathematics, Thm.3.6.9)

Filter pairs and Galois connections

Corollary:
Let (G, i) be a filter pair. Then i has a left adjoint Ξ.

$$\begin{tikzcd} G(A) \arrow[r, phantom, "\perp"] \arrow[r, bend right, "i_A"'] & \mathcal{P}(A) \arrow[l, bend right, "\Xi_A"'] \end{tikzcd}$$

The closure operator of the associated logic L is $i_{Fm_\Sigma(X)} \circ \Xi_{Fm_\Sigma(X)}$.

More generally, for every algebra A we get two g-matrices, $\langle A, Fi_L(A)\rangle$ and $\langle A, \{(i_A \circ \Xi_A)\text{-closed sets}\}\rangle$.

Consider an equational filter pair (Co_K, i):
The left adjoint to $i\colon Co_K \to \wp(A)$, $\theta \mapsto \{a \mid \langle \delta(a), \epsilon(a) \rangle \in \theta\}$ is given by

$$\Xi(F) = [\text{the } K\text{-congruence generated by } \{\langle \delta(a), \epsilon(a) \rangle \mid a \in F\}]$$

The associated logic L is protoalgebraic iff Ω is monotonous iff Ω preserves arbitrary infima iff Ω has a left adjoint ℓ

$$\begin{aligned}
\ell(\theta) &= \bigwedge\{F \mid \theta \subseteq \Omega(F)\} \\
&= \bigcap\{F \mid F \text{ is union of } \theta\text{-equivalence classes}\} \\
&= \bigcup\{\theta\text{-equivalence classes of theorems of } l\} \\
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(Compare: If the equation is given by $\langle, \top\rangle$, then $i(\theta) = [\top]_\theta$)

Proposition:

Let (Co_K, i) be an equational filter pair. Let A be a Σ-algebra and $\theta \in Co_K(A)$ a congruence on A. Then $\Xi_A(i(\theta)) \subseteq \theta \subseteq \Omega_A(i_A(\theta))$

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Let (Co_K, i) be an equational filter pair. Let A be a Σ-algebra and $\theta \in Co_K(A)$ a congruence on A. Then $\Xi_A(i(\theta)) \subseteq \theta \subseteq \Omega_A(i_A(\theta))$

Proposition: The following are equivalent:

(i) Both of the inclusions are equalities

(ii) One of the inclusions is an equality

(iii) i is injective

If these conditions hold, then:

- L is algebraizable, with associated quasivariety K
- $\Omega = \Xi = i^{-1}$

Vista

A. Further bridge theorems like Craig interpolation.

B. Infinitary version (work in progress)

C. Further examples of filter pairs (e.g. relation to non-deterministic semantics)