Generalized bunched implication algebras
(Dedicated to the memory of Bjarni Jónsson)

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Structure of the talk

- Motivation and examples
- Algebraic Theory
- Proof Theory

Outline

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- Motivation and examples
- Algebraic Theory
- Proof Theory

Bunched Implication Logic

- Motivated by separation logic used in pointer management in computer science.
- It is a substuctural logic and it combines an additive (Heyting) implication and a multiplicative (linear) implication.

Residuated lattices

A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
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$$ab \leq c \;\Leftrightarrow\; b \leq a\backslash c \;\Leftrightarrow\; a \leq c/b.$$

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In every residuated lattice multiplication disctributes over join, so in a Heyting algebra the lattice is distributive.

In general the lattice reduct need not be distributive, as in the lattice of ideals of a ring.
$I \wedge J = I \cap J$,
$I \vee J = I + J$, and
IJ contains finite sums of products ij, as usual.

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- BL-algebras
- Lattice-ordered groups
- Relation algebras

GBI algebras

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- Relation algebras

A *Generlized Bunched Implication algebra* (or *GBI algebra*) $\mathbf{A} = (A \wedge, \vee, \cdot, \backslash, /, 1, \to, \top)$ supports two residuated structures: a residuated lattice $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ and a Browerian/Heyting algebra $(A, \wedge, \vee, \to, \top)$.

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Given a set P for binary relations $R, S \in \mathcal{P}(P \times P)$, we define

- $R \wedge S = R \cap S$
- $R \vee S = R \cup S$
- $R \cdot S = R \circ S$ (relational composition)
- $R \to S = R^c \cup S = (R \cap S^c)^c$
- $R \backslash S = (R^\cup \circ S^c)^c$ (where $R^\cup$ is the converse of R)
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This is an example of a GBI algebra, and part of is special nature is the fact that the Heyting algebra reduct is actually Boolean. We consider generalizations of these algebras called weakening relation algebras.

Weakening relation algebras

Instead of a set P we begin with a poset $\mathbf{P} = (P, \leq)$. (We could recover the previous case by taking the discrete order.)

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We define the set $Wk(\mathbf{P})$ of $\leq$-*weakening relations*, that is of all binary relations R on P such that $a \leq b \; R \; c \leq d$ implies $a \; R \; d$, for all $a, b, c, d \in P$.

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We now explain why $Wk(\mathbf{P})$ supports a structure of a GBI-algebra, under union and intersection, and composition of relations.

Conuclei

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A *topological weak conucleus* further satisfies $\sigma(x) \wedge \sigma(y) \leq \sigma(x \wedge y)$. So, a topological weak conucleus on a GBI-algebra $\mathbf{A}$ is a weak conucleus on both the residuated lattice and the Brouwerian algebra reducts of $\mathbf{A}$.

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Given a residuated lattice $\mathbf{A}$ and a positive idempotent element p, the map σ_p, where $\sigma_p(x) = p\backslash x/p$, is a topological weak conucleus called the *double division conucleus by* p.

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Given a poset $\mathbf{P} = (P, \leq)$, we set $\mathbf{A} = Rel(P)$, to be the involutive GBI algebra of all binary relations on the set P. Note that $p = \leq$ is a positive idempotent element of $\mathbf{A}$. It is easy to see that $p\backslash\mathbf{A}/p$ is exactly $Wk(\mathbf{P})$, so the latter is a GBI-algebra.

WK as a conucleus image

If $\mathbf{A}$ is involutive then so is $p\backslash\mathbf{A}/p$ and the latter is a subalgebra of $\mathbf{A}$ with respect to the operations $\wedge, \vee, \cdot, +, \sim, -$.

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We note that we also have that $Wk(\mathbf{P}) \cong Res(\mathcal{O}(\mathbf{P}))$. Recall that for a complete join semilattice $\mathbf{L}$, $Res(\mathbf{L})$ denotes the residuated lattice of all residuated maps on $\mathbf{L}$; here a map on f on a poset $\mathbf{P}$ is called *residuated* if there exists a map f^* on P such that $f(x) \leq y$ iff $x \leq f^*(y)$, for all $x, y \in P$.

Algebraic Theory

The study of congruences of the algebraic models is important in determining subdirectly irreducibles, subvarieties, deduction theorems. We prove that congruences on an algebra correspond to specific subsets. As in the case of group theory (normal subgroups) this proves to be a substantial simplification.

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It is known that if θ is a congruence on $\mathbf{A}$ then $\uparrow[1]_\theta$, the upset of the equivalence class of 1, is a normal multiplicative filter. Conversely, if F is a normal multiplicative filter of a residuated lattice $\mathbf{A}$, then the relation θ_F is a congruence on $\mathbf{A}$, where $a \, \theta_F \, b$ iff $a\backslash b \wedge b\backslash a \in F$.

Alternative subsets to F include convex normal (for $\rho_a x = (ax/a) \wedge 1$ and $\lambda_a(x) = (a\backslash xa) \wedge 1$))subalgebras, such as $\{x : f \leq x \leq 1/f, f \in F\}$ and also convex normal (for ρ, λ) negative submonoids, such as the negative cone of F: $\{x \in F : x \leq 1\}$.

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GBI-congruences are RL-congruences with further closure conditions. As a result the equivalence class of 1 is a normal multiplicative filter with further closure conditions. We identify these as closure under
$r_{a,b}(x) = (a \to b)/(xa \to b)$ and
$s_{a,b}(x) = (a \to bx)/(a \to b)$, for all a, b.

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Alternatively, congruences are characterized by their equivalence classes of $\top$. These are usual lattice filters that are closed under
$u_{a,b}(x) = a/(b \wedge x) \to a/b$,
$u'_{a,b}(x) = (b \wedge x)\backslash a \to b\backslash a$,
$v_{a,b}(x) = ab \to (a \wedge x)b$,
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$w(x) = \top\backslash x/\top$, for all a, b.

As a result we obtain a parameterized local deduction theorem for the GBI.

Proof Theory

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We consider the set of GBI-formulas Fm and define the free algebra W over Fm with two operations $\circ$ (also denoted by comma) and $\owedge$ (also denoted by semicolon). A sequent (also called a bunch) is an expression of the form $x \Rightarrow a$, where $x \in W$ and $a \in Fm$. For example,

$$(q \owedge (p \to r)) \circ (p \cdot q) \Rightarrow (p \to q)\backslash(q \wedge r)$$

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We will consider extensions by any equations over the signature $\{\vee, \wedge, \cdot, 1\}$ of this calculus and study cut elimination, decidability, finite model property, finite embeddability property.

The Gentzen calculus

$$\frac{x \Rightarrow a \quad u(a) \Rightarrow c}{u(x) \Rightarrow c}\ (\mathsf{CUT}) \qquad \frac{}{a \Rightarrow a}\ (\mathsf{Id}) \qquad \frac{u(x \bigcirc\!\!\!\!\wedge (y \bigcirc\!\!\!\!\wedge z)) \Rightarrow c}{\overline{u((x \bigcirc\!\!\!\!\wedge y) \bigcirc\!\!\!\!\wedge z) \Rightarrow c}}\ (\bigcirc\!\!\!\!\wedge\, a)$$

$$\frac{u(x \bigcirc\!\!\!\!\wedge y) \Rightarrow c}{u(y \bigcirc\!\!\!\!\wedge x) \Rightarrow c}\ (\bigcirc\!\!\!\!\wedge\, e) \qquad \frac{u(x) \Rightarrow c}{u(x \bigcirc\!\!\!\!\wedge y) \Rightarrow c}\ (\bigcirc\!\!\!\!\wedge\, i) \qquad \frac{u(x \bigcirc\!\!\!\!\wedge x) \Rightarrow c}{u(x) \Rightarrow c}\ (\bigcirc\!\!\!\!\wedge\, c)$$

$$\frac{u(a) \Rightarrow c \quad u(b) \Rightarrow c}{u(a \vee b) \Rightarrow c}\ (\vee\mathsf{L}) \qquad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee\mathsf{R}\ell) \qquad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee\mathsf{R}r)$$

$$\frac{u(a \bigcirc\!\!\!\!\wedge b) \Rightarrow c}{u(a \wedge b) \Rightarrow c}\ (\wedge\mathsf{L}) \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge\mathsf{R})$$

$$\frac{u(a \circ b) \Rightarrow c}{u(a \cdot b) \Rightarrow c}\ (\cdot\mathsf{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot\mathsf{R}) \qquad \frac{u(\varepsilon) \Rightarrow a}{u(1) \Rightarrow a}\ (1\mathsf{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathsf{R})$$

$$\frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u(x \circ (a\backslash b)) \Rightarrow c}\ (\backslash\mathsf{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a\backslash b}\ (\backslash\mathsf{R}) \qquad \frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u((b/a) \circ x) \Rightarrow c}\ (/\mathsf{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathsf{R})$$

$$\frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u(x \bigcirc\!\!\!\!\wedge (a \to b)) \Rightarrow c}\ (\to\mathsf{L}) \qquad \frac{x \bigcirc\!\!\!\!\wedge a \Rightarrow b}{x \Rightarrow a \to b}\ (\to\mathsf{R}) \qquad \frac{u(\delta) \Rightarrow c}{u(\top) \Rightarrow c}\ (\top\mathsf{L}) \qquad \frac{}{x \Rightarrow \top}\ (\top\mathsf{R})$$

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The theory of residuated frames was developed in some earlier work and it is extended to the GBI setting here.

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We consider further structural rules of the following form, where $t_0, t_1, \ldots, t_n \in W$ (and no variables are repeated in t_0).

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Decidability

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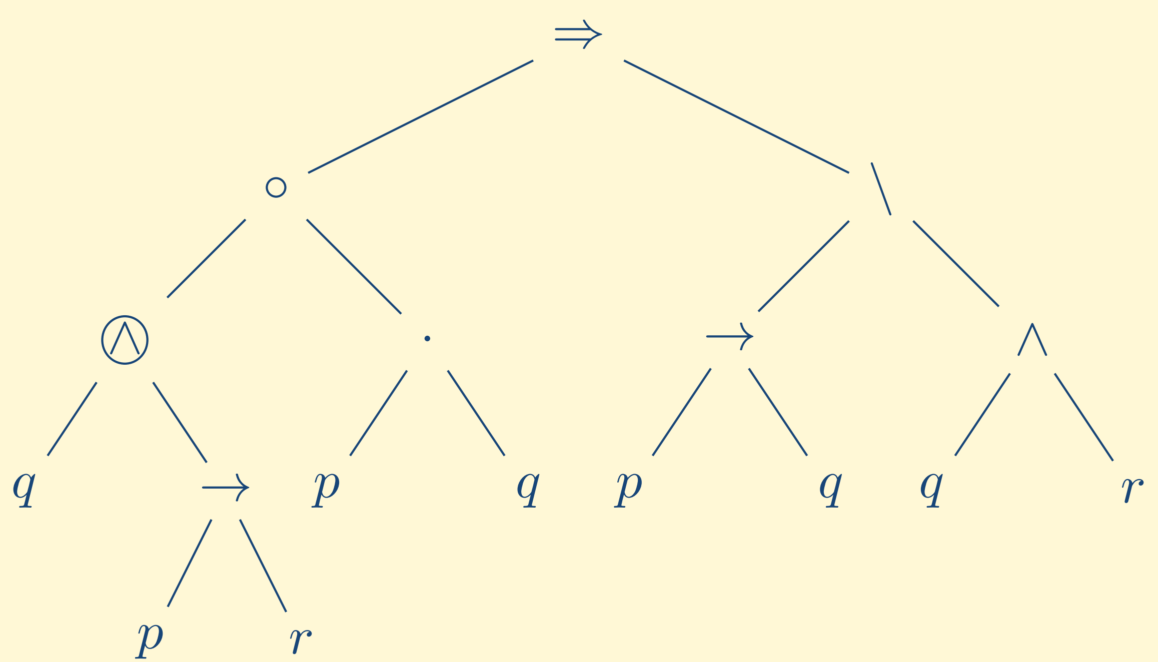
$$(q \owedge (p \to r)) \circ (p \cdot q) \Rightarrow (p \to q)\backslash(q \wedge r)$$



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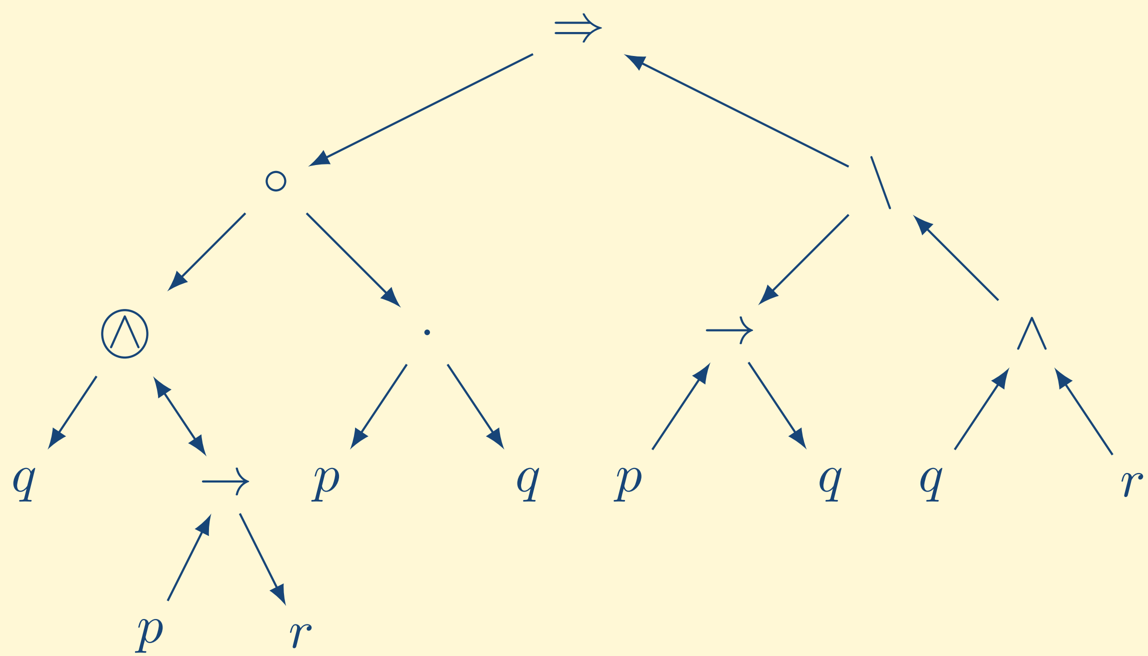
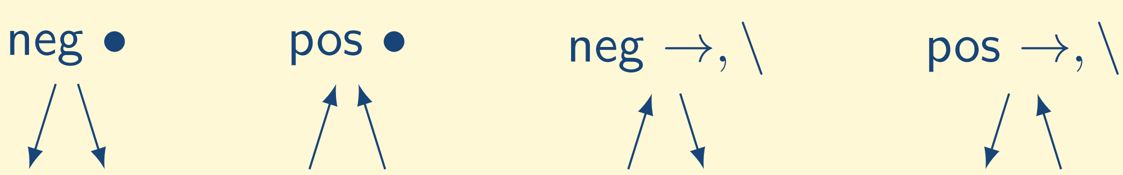


The two edges below a $\circ$ or a $\owedge$ point downward (and the same holds for the connectives $\wedge$, $\vee$ and $\cdot$ in *negative position*). Here $\bullet$ is any of $\circ, \owedge, \cdot, \wedge, \vee$.



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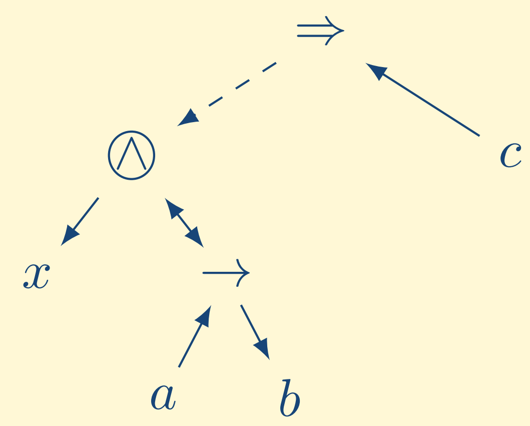

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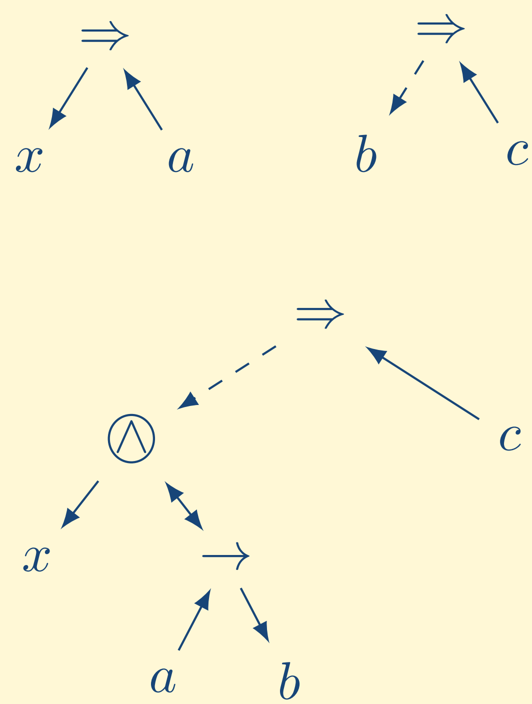

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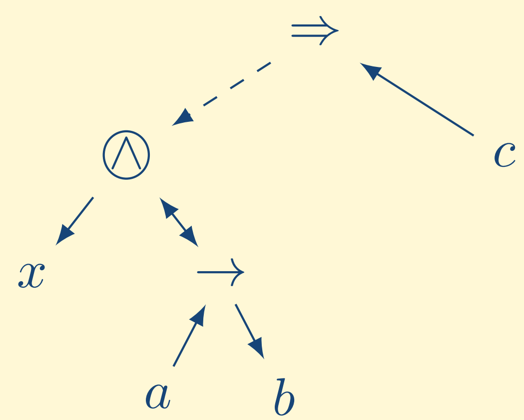

This puts a bound on the $\circ$-tree height of all sequents in the proof of a sequent. Also, since we can restrict to proofs of 3-reduced sequents, this supports an inductive argument of finiteness.

FMP

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Even though the proof search of s is infinite, we argue that $\mathbf{W}^+$ is finite and refutes s.

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Then using well quasiorders and better quasiorders we can show that $\mathbf{W}^+$ is finite for many subvarieties. [Joint work with Riquelmi Cardona]