Generalized bunched implication algebras
(Directed to the memory of Bjarni Jónsson)

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Structure of the talk

- Motivation and examples
- Algebraic Theory
- Proof Theory
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Bunched Implication Logic

- Motivated by separation logic used in pointer management in computer science.
- It is a substructural logic and it combines an additive (Heyting) implication and a multiplicative (linear) implication.
A *residuated lattice*, is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \backslash, /, 1)$ such that

- $(L, \wedge, \vee)$ is a lattice,
- $(L, \cdot, 1)$ is a monoid and
- for all $a, b, c \in L$,

$$ab \leq c \iff b \leq a \backslash c \iff a \leq c / b.$$
A **residuated lattice**, is an algebra \( L = (L, \land, \lor, \cdot, \backslash, /, 1) \) such that

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\[
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\]

If \( xy = x \land y \) then \( L \) is a **Brouwerian algebra** (Heyting algebra, if there is a bottom element). In this case we write \( x \rightarrow y \) for \( x \backslash y = y / x \).
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If $xy = x \wedge y$ then $\mathbb{L}$ is a **Brouwerian algebra** (Heyting algebra, if there is a bottom element). In this case we write $x \rightarrow y$ for $x \backslash y = y / x$.

In every residuated lattice multiplication distributes over join, so in a Heyting algebra the lattice is distributive.
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In general the lattice reduct need not be distributive, as in the lattice of ideals of a ring.

$I \wedge J = I \cap J$,
$I \vee J = I + J$, and
$IJ$ contains finite sums of products $ij$, as usual.
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- BL-algebras
- Lattice-ordered groups
- Relation algebras
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A *Generalized Bunched Implication algebra* (or *GBI algebra*)

\[ \mathbf{A} = (A \wedge, \vee, \cdot, \backslash, /, 1, \to, \top) \]

supports two residuated structures: a residuated lattice \((A, \wedge, \vee, \cdot, \backslash, /, 1)\) and a Browerian/Heyting algebra \((A, \wedge, \vee, \to, \top)\).
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Given a set $P$ for binary relations $R, S \in \mathcal{P}(P \times P)$, we define

- $R \land S = R \cap S$
- $R \lor S = R \cup S$
- $R \cdot S = R \circ S$ (relational composition)
- $R \to S = R^c \cup S = (R \cap S^c)^c$
- $R \setminus S = (R^c \circ S^c)^c$ (where $R^c$ is the converse of $R$)
- $S/R = (S^c \circ R^c)^c$
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This is an example of a GBI algebra, and part of is special nature is the fact that the Heyting algebra reduct is actually Boolean. We consider generalizations of these algebras called weakening relation algebras.
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We define the set $Wk(P)$ of $\leq$-weakening relations, that is of all binary relations $R$ on $P$ such that $a \leq b \mathrel{R} c \leq d$ implies $a \mathrel{R} d$, for all $a, b, c, d \in P$. 
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Weakening relation algebras

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We now explain why $Wk(P)$ supports a structure of a GBI-algebra, under union and intersection, and composition of relations.
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A weak conucleus on a residuated lattice $\mathbf{A}$ is an interior operator $\sigma$ on $\mathbf{A}$ such that $\sigma(x) \sigma(y) \leq \sigma(xy)$, for all $x, y \in \mathbf{A}$. Then $\sigma[\mathbf{A}] = (\sigma[\mathbf{A}], \wedge, \vee, \cdot, \backslash, / \sigma)$ is a residuated lattice-ordered semigroup, where $x \cdot_{\sigma} y = \sigma(x \cdot y)$, where $\cdot \in \{\wedge, \backslash, /\}$. 
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A topological weak conucleus further satisfies $\sigma(x) \land \sigma(y) \leq \sigma(x \land y)$. So, a topological weak conucleus on a GBI-algebra $A$ is a weak conucleus on both the residuated lattice and the Brouwerian algebra reducts of $A$. 
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Given a residuated lattice $\mathbf{A}$ and a positive idempotent element $p$, the map $\sigma_p$, where $\sigma_p(x) = p \backslash x / p$, is a topological weak conucleus called the double division conucleus by $p$. 
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Nick Galatos, TACL, Prague, June 2017
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Given a poset $P = (P, \leq)$, we set $A = Rel(P)$, to be the involutive GBI algebra of all binary relations on the set $P$. Note that $p = \leq$ is a positive idempotent element of $A$. It is easy to see that $p \setminus A / p$ is exactly $Wk(P)$, so the latter is a GBI-algebra.
If $A$ is involutive then so is $p \setminus A / p$ and the latter is a subalgebra of $A$ with respect to the operations $\land, \lor, \cdot, +, \sim, \neg$. 
If $A$ is involutive then so is $p \setminus A / p$ and the latter is a subalgebra of $A$ with respect to the operations $\wedge, \vee, \cdot, +, \sim, -$. Recall that an involutive residuated lattice is an expansion of a residuated lattice with an extra constant $0$ such that $\sim(-x) = x = -(\sim x)$, where $\sim x = x \setminus 0$ and $-x = 0 / x$; we also define $x + y = \sim(-y \cdot -x)$.
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We note that we also have that $Wk(P) \cong Res(O(P))$. Recall that for a complete join semilattice $L$, $Res(L)$ denotes the residuated lattice of all residuated maps on $L$; here a map on $f$ on a poset $P$ is called residuated if there exists a map $f^*$ on $P$ such that $f(x) \leq y$ iff $x \leq f^*(y)$, for all $x, y \in P$. 
The study of congruences of the algebraic models is important in determining subdirectly irreducibles, subvarieties, deduction theorems. We prove that congruences on an algebra correspond to specific subsets. As in the case of group theory (normal subgroups) this proves to be a substantial simplification.
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In residuated lattices congruences correspond to *normal submonoid filters*. Given $a, x \in A$ we define $\rho'_a x = ax/a$ and $\lambda'_a(x) = a \setminus xa$ (which are akin to conjugates in group theory). A subset is called *normal* if it is closed under $\rho'_a$ and $\lambda'_a$ for all $a \in A$. 
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It is known that if $\theta$ is a congruence on $A$ then $\uparrow \{1\}_\theta$, the upset of the equivalence class of 1, is a normal multiplicative filter. Conversely, if $F$ is a normal multiplicative filter of a residuated lattice $A$, then the relation $\theta_F$ is a congruence on $A$, where $a \theta_F b$ iff $a \setminus b \land b \setminus a \in F$. 
Alternative subsets to $F$ include convex normal (for $\rho_a x = (ax/a) \land 1$ and $\lambda_a (x) = (a \backslash xa) \land 1$) subalgebras, such as $\{x : f \leq x \leq 1/f, f \in F\}$ and also convex normal (for $\rho, \lambda$) negative submonoids, such as the negative cone of $F$: $\{x \in F : x \leq 1\}$. 
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Note that if $A$ is a Brouwerian or a Heyting algebra, then all notions
coincide: normal multiplicative filters, convex normal subalgebras,
and convex normal negative submonoids, are usual lattice filters.
Alternative subsets to $F$ include convex normal (for $\rho_a x = (ax/a) \land 1$ and $\lambda_a(x) = (a \setminus xa) \land 1$) subalgebras, such as $\{x : f \leq x \leq 1/f, f \in F\}$ and also convex normal (for $\rho, \lambda$) negative submonoids, such as the negative cone of $F$: $\{x \in F : x \leq 1\}$.

Note that if $A$ is a Brouwerian or a Heyting algebra, then all notions coincide: normal multiplicative filters, convex normal subalgebras, and convex normal negative submonoids, are usual lattice filters.

GBI-congruences are RL-congruences with further closure conditions. As a result the equivalence class of 1 is a normal multiplicative filter with further closure conditions. We identify these as closure under $r_{a,b}(x) = (a \rightarrow b)/(xa \rightarrow b)$ and $s_{a,b}(x) = (a \rightarrow bx)/(a \rightarrow b)$, for all $a, b$. 
Alternatively, congruences are characterized by their equivalence classes of $\top$. These are usual lattice filters that are closed under

\begin{align*}
  u_{a,b}(x) &= a/(b \land x) \rightarrow a/b, \\
  u'_{a,b}(x) &= (b \land x)\!\backslash\!a \rightarrow b\!\backslash\!a, \\
  v_{a,b}(x) &= ab \rightarrow (a \land x)b, \\
  v'_{a,b}(x) &= ab \rightarrow a(b \land x), \text{ and} \\
  w(x) &= \top\!\backslash\!x/\top, \text{ for all } a, b.
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  w(x) &= \top \backslash x / \top, \text{ for all } a, b.
\end{align*}
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As a result we obtain a parameterized local deduction theorem for the GBI.
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We consider the set of GBI-formulas $Fm$ and define the free algebra $W$ over $Fm$ with two operations $\circ$ (also denoted by comma) and $\otimes$ (also denoted by semicolon). A sequent (also called a bunch) is an expression of the form $x \Rightarrow a$, where $x \in W$ and $a \in Fm$. For example,

$$(q \otimes (p \rightarrow r)) \circ (p \cdot q) \Rightarrow (p \rightarrow q) \backslash (q \land r)$$
Starting from GBI-algebras we can present a display calculus for it, in a natural way. However, a standard *Genzen-style* formalism also enjoys enough display properties and is simpler. The following calculus is well known, starting from the relevance logic community.

We consider the set of GBI-formulas \( F_m \) and define the free algebra \( W \) over \( F_m \) with two operations \( \circ \) (also denoted by comma) and \( \ominus \) (also denoted by semicolon). A sequent (also called a bunch) is an expression of the form \( x \Rightarrow a \), where \( x \in W \) and \( a \in F_m \). For example,

\[(q \ominus(p \rightarrow r)) \circ (p \cdot q) \Rightarrow (p \rightarrow q) \backslash (q \land r)\]

We will consider extensions by any equations over the signature \( \{\lor, \land, \cdot, 1\} \) of this calculus and study cut elimination, decidability, finite model property, finite embeddability property.
The Gentzen calculus

\[
\frac{x \Rightarrow a \quad u(a) \Rightarrow c}{u(x) \Rightarrow c} \quad \text{(CUT)} \quad \frac{u(x) \Rightarrow c}{u(x) \Rightarrow c} \quad \frac{u(x \bigvee (y \bigwedge z)) \Rightarrow c}{u((x \bigvee y) \bigwedge z) \Rightarrow c} \quad \frac{u(x \bigwedge y) \Rightarrow c}{u(x \bigwedge x) \Rightarrow c} \quad \text{(\bigwedge i)} \quad \frac{u(x \bigwedge x) \Rightarrow c}{u(x) \Rightarrow c} \quad \text{(\bigwedge c)}
\]

\[
\frac{u(a) \Rightarrow c \quad u(b) \Rightarrow c}{u(a \bigvee b) \Rightarrow c} \quad \text{(\bigvee L)} \quad \frac{x \Rightarrow a}{x \Rightarrow a \bigvee b} \quad \text{(\bigvee R)} \quad \frac{x \Rightarrow b}{x \Rightarrow a \bigvee b} \quad \text{(\bigvee Rr)} \quad \frac{u(a \bigwedge b) \Rightarrow c}{u(a \bigwedge a) \Rightarrow c} \quad \text{(\bigwedge L)} \quad \frac{u(a \bigwedge b) \Rightarrow c}{u(a \bigwedge b) \Rightarrow c} \quad \text{(\bigwedge R)}
\]

\[
\frac{u(a \circ b) \Rightarrow c}{u(a \cdot b) \Rightarrow c} \quad \text{(.L)} \quad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \Rightarrow a \quad y \Rightarrow b} \quad \text{(.R)} \quad \frac{u(\varepsilon) \Rightarrow a}{u(1) \Rightarrow a} \quad \text{(1L)} \quad \frac{\varepsilon \Rightarrow 1}{1R}
\]

\[
\frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u(x \bigcirc (a \backslash b)) \Rightarrow c} \quad \text{(/L)} \quad \frac{a \bigcirc x \Rightarrow b}{x \Rightarrow a \backslash b} \quad \text{(/R)} \quad \frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u((b/a) \bigcirc x) \Rightarrow c} \quad \text{(\backslash L)} \quad \frac{x \bigcirc a \Rightarrow b}{x \Rightarrow b/a} \quad \text{(\backslash R)}
\]

\[
\frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u(x \bigotimes (a \rightarrow b)) \Rightarrow c} \quad \text{(-L)} \quad \frac{x \bigotimes a \Rightarrow b}{x \Rightarrow a \rightarrow b} \quad \text{(-R)} \quad \frac{u(\delta) \Rightarrow c}{u(\top) \Rightarrow c} \quad \text{(\top L)} \quad \frac{x \Rightarrow \top}{x \Rightarrow \top} \quad \text{(\top R)}
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We define the relation $N$ between $W$ and $Fm$ by writing $x N a$ if the sequent $x \Rightarrow a$ is cut-free provable. This then supports the structure of a GBI-frame $W = (W, \circ, \sqcap, N, Fm)$ and it yields a GBI-algebra $W^+$; it can be shown that this algebra that refutes any non-provable sequent.

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We define the relation $N$ between $W$ and $Fm$ by writing $x N a$ if the sequent $x \Rightarrow a$ is cut-free provable. This then supports the structure of a GBI-frame $W = (W, \circ, \bigwedge, N, Fm)$ and it yields a GBI-algebra $W^+$; it can be shown that this algebra that refutes any non-provable sequent.

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We consider further structural rules of the following form, where $t_0, t_1, \ldots, t_n \in W$ (and no variables are repeated in $t_0$).

$$
\begin{array}{c}
\frac{u(t_1) \Rightarrow a \quad \cdots \quad u(t_n) \Rightarrow a}{u(t_0) \Rightarrow a}
\end{array} \quad [r]
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We can prove that if we add $[r]$ to the calculus then the algebra $W^+$ satisfies the identity $t_0 \leq t_1 \lor \cdots \lor t_n$. This yealds cut elimination for all such extensions in the signature $\{\lor, \land, \cdot, 1\}$.
Decidability

Given a sequent $x \Rightarrow a$ we define its *sequent tree* (growing downward) in the obvious way:
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Given a sequent $x \Rightarrow a$ we define its sequent tree (growing downward) in the obvious way: $\Rightarrow$ sits the root with two children nodes; on the right-node sits the formula tree of $a$; on the left-node sits the structure tree of $x$. For example we can take the sequent

$$(q \sqcap (p \rightarrow r)) \circ (p \cdot q) \Rightarrow (p \rightarrow q) \setminus (q \land r)$$
We now add *directions* to the edges of this tree.
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The two edges below a ☐ or a \( \& \) point downward (and the same holds for the connectives \( \& \), \( \lor \) and \( \cdot \) in *negative position*). Here \( \bullet \) is any of ☐, \( \& \), \( \cdot \), \( \& \), \( \lor \).
The *multiplicative length* of a sequent is defined along an oriented path by counting the maximum numbers of $\circ$, $\cdot$ in negative position and of $\setminus$, $/$ in positive position. Note that the multiplicative length does not increase upwards by the rules. Care is needed for $(\rightarrow L)$:
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\[
\Rightarrow \\
\begin{array}{c}
 x \\
 a \\
 b \\
 c
\end{array} \\
\Rightarrow \\
\begin{array}{c}
 \land \\
 x \\
 a \\
 b \\
 \Rightarrow \\
 c
\end{array}
\]

This puts a bound on the $\circ$-tree height of all sequents in the proof of a sequent.
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![Diagram of sequents](image)

This puts a bound on the $\circ$-tree height of all sequents in the proof of a sequent. Also, since we can restrict to proofs of 3-reduced sequents, this supports an inductive argument of finiteness.
To show the Finite Model Property we start with a sequent $s$ that is not provable and construct a finite countermodel. We modify $W$, since $W^+$ was infinite.
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Even though the proof search of $s$ is infinite, we argue that $W^+$ is finite and refutes $s$. 
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We modify the frame by taking $W$ to be the subset of $A$ generated by $B$ using multiplication and meet. Also, for $x \in W$ and $b \in B$, we define $x \ll b$ iff $x \leq b$. 
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Then using well quasiorders and better quasiorders we can show that $W^+$ is finite for many subvarieties. [Joint work with Riquelmi Cardona]