

Many-Valued Arrow Logic with Scalar Multiplication

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Introduction

- 1 The problem
- 2 MALS
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- 4 Scalar operations
- 5 Future work

Basic Arrow Logic (BAL): Syntax

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Rule:

$\forall c_i \in \mathcal{C}$, and $\forall p_i \in \mathcal{P}$:

$\varphi ::= A \mid \neg A \mid \otimes A \mid A \wedge B \mid A \vee B \mid A \supset B \mid A \equiv B \mid A \circ B \mid f$

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$\mathbb{M}, a, \models A \iff a \in v(A)$

$\mathbb{M}, a, \models \neg A \iff \mathbb{M}, a, \not\models A$

$\mathbb{M}, a, \models \otimes A \iff \exists b \text{ s.t. } Rab \ \& \ \mathbb{M}, b, \models A$

$\mathbb{M}, a, \models A \wedge B \iff \mathbb{M}, a, \models A \ \& \ \mathbb{M}, a, \models B$

$\mathbb{M}, a, \models A \vee B \iff \mathbb{M}, a, \models A \text{ or } \mathbb{M}, a, \models B$

$\mathbb{M}, a, \models A \circ B \iff \exists b, c \text{ s.t. } Cab \ \& \ \mathbb{M}, b, \models A \ \& \ \mathbb{M}, c, \models B$

$\mathbb{M}, a, \models \int \iff Ia$

$(A \supset B) =_{df} (\neg A \vee B)$

$(A \equiv B) =_{df} ((A \supset B) \wedge (B \supset A))$

$\Gamma \models \varphi \iff \mathbb{M}, a, \models A \text{ (for all } A \in \Gamma\text{), then } \mathbb{M}, a, \models \varphi$

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Vector Spaces Axioms:

$$a + b = b + a$$

$$a + (b + c) = (a + b) + c$$

$$a + 0 = a$$

$$a + (-a) = 0$$

$$0 \cdot a = 0$$

$$1 \cdot a = a$$

$$x \cdot (y \cdot a) = (x \cdot y) \cdot a$$

$$(x + y)a = (x \cdot a) + (y \cdot a)$$

$$x(a + b) = (x \cdot a) + (x \cdot b)$$

Possible problem: Scalars

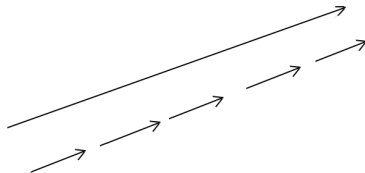


Figura: Multiplication

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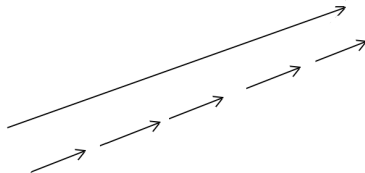


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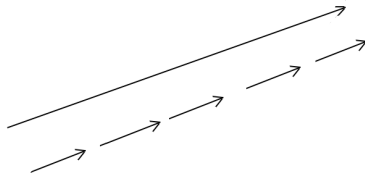


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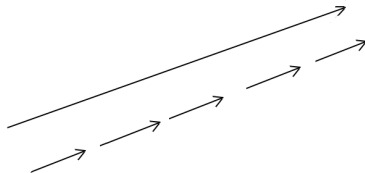


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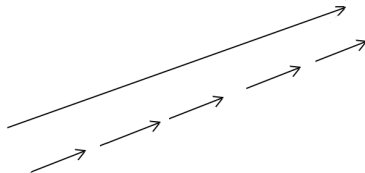


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Limitations: C-decomposition works only with positive integers

Our approach extends this definition

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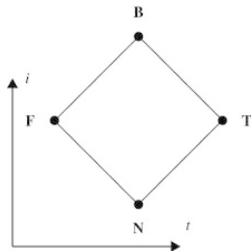


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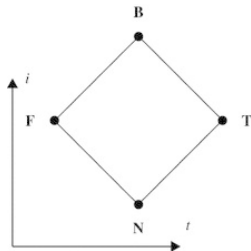


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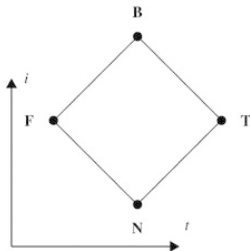


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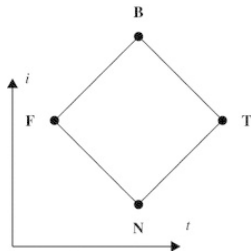


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$$v : For \times I \longrightarrow V$$

$$v_i(\Box A) = \inf\{v_j(A) : Rij\}$$

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A set of arrows,

$C_* \subseteq A \times A^n, \langle i, \langle j_1, \langle \dots, j_n \rangle \rangle \rangle \in C_*$

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Logical consequence: $\models \subseteq P(\text{Form}) \longrightarrow \text{Form}$

$$\Gamma \models \varphi \iff \forall \beta \in \Gamma, v_i(\beta) \in D^+ \Rightarrow v_i(\varphi) \in D^+$$

Examples of problematic inferences

$$(A \circ B), \neg(A \circ B) \models_{BAL} C$$

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Proof:

Suppose $(A \circ B), \neg(A \circ B) \not\models_{BAL} C$

$v_i(A \circ B) = (\neg(A \circ B)) = 1$, and $v_i(C) = 0$

If $v_i(A \circ B) = 1$ then $\exists j, k \in A, Cijk : v_j(A) = 1$ and $v_k(B) = 1$

If $v_i(\neg(A \circ B)) = 1$ then $v_i(A \circ B) = 0$, therefore

$\forall jk \in A, Cijk : v_j(A) = 0$ or $v_k(B) = 0$

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We add

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We add

$$*_1 A \parallel A$$

$$*_0 A \parallel \int$$

$$*_m *_n A \parallel *_n *_m A$$

$$*_{n+m} A \parallel (*_n A \circ *_m A)$$

$$*_n (A \circ B) \parallel (*_n A \circ *_n B)$$

Two examples

$$\begin{aligned} *_{\frac{1}{2}} *_{\frac{1}{2}} A &\parallel A \\ *_{-1} *_4 A &\parallel *_3 A \end{aligned}$$

future work

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