Algebra
Clones on finite sets

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We will consider "operations" on a fixed set M, usually a finite set with at least two elements. An operation is a total function f from M^k to M, for some natural number k, the "arity" of f. Typical and already nontrivial examples are Boolean operations, which are operations on the 2-element set true, false , such as conjunction or negation. Operations on larger sets can be viewed as logical operation in a multivalued logic.

A "clone" C is a set of operations on a fixed set M which is closed under the natural notion of composition, and moreover contains all the projection operations such as $f(x, y, z) = z$. Typical examples of clones are: the set of polynomials on a ring R, or the set of all (pointwise) monotone operations on a partial order P.

The smallest clone on M is the clone of all projections, which is too small to be useful for almost anything; the largest clone is the clone of all operations on M, which is usually to unwieldy to work with. We will be interested in the set $Cl(M)$ of ALL clones on a given set M; equipped with the subset relation this set becomes a complete lattice (a partial order in which every set has a least upper and a greatest lower bound). While the structure of this lattice is too complicated to be fully understood, there are still many results which give a partial classification of clones, or of large subsets of $Cl(M)$.

A clone that consists of unary functions only is a subsemigroup (even submonoid) of the transformation monoid of all unary functions $f\colon M \to M$. The well-known relationship between transformation monoids (with a concrete composition operation) and abstract monoids, or similarly the relationship between permutation groups and abstract groups can be generalized to clones with operations of higher arity. In particular, we can define an abstract clone to be an algebra with an abstract "composition" operation which satisfies all the higher associativity laws that every concrete clone satisfies; an analog of Cayley's representation theorem shows that every abstract clone will be isomorphic to a concrete clone.

In my lectures I plan to present the following topics:

- Definition of clones.
- examples: from below, from above
- the Galois connection between functions and relations
- finitely vs infinitely generated clones
- decidability issues (is f in C? Is C a subset of D?)
- clones on the 2-element set. Post's lattice.
- the clone lattice on finite sets of size > 2
- maximal clones (co-atoms) as a completeness criterion
- Rosenberg's list of maximal clones on any finite set.
- remarks on clones on infinite sets
- results on minimal clones
- interpolation of operations by operations from a clone: on k points, on finitely many points, k-closure, local closure of clones
- intervals in the clone lattice: simple and complicated, intervals below a given clone, intervals containing a given clone
- abstract clones, representation by concrete clones