The Variety of Nuclear Implicative Semilattices is Locally Finite

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We study the variety of *nuclear algebras* — algebras of the form $(A, \wedge, 1, \to, \mathbf{j})$ where $(A, \wedge, 1, \to)$ is a meet-implicative semilattice and $\mathbf{j} : A \to A$ is a *nucleus*.

The latter means that $\mathbf{j}$ is an idempotent inflationary multiplicative unary operator, that is, the identities

$$\begin{aligned} &x \leqslant \mathbf{j}x \\ &\mathbf{jj}x = \mathbf{j}x \\ &\mathbf{j}(x \wedge y) = \mathbf{j}x \wedge \mathbf{j}y \end{aligned}$$

hold in A.

It has been proved by Diego in [1] that the variety of meet-implicative semilattices is locally finite. Our main result is that the same remains true after extending the signature with a nucleus as above.

Archetypal example of an implicative semilattice: let $(X, \leqslant)$ be a poset (partially ordered set), and let $A = \mathbf{D}(X, \leqslant)$ be the set of *downsets* of $(X, \leqslant)$ (subsets $D \subseteq X$ satisfying $x \in D$, $y \leqslant x \Rightarrow y \in D$ for all $x, y \in X$). Let us equip A with the semilattice structure via $D_1 \wedge D_2 := D_1 \cup D_2$; it has unit $1 := \varnothing$ and the implication given by $D_1 \to D_2 := \downarrow(D_2 - D_1)$, where for a subset $S \subseteq X$, we denote by $\downarrow(S)$ the smallest downset containing S, i. e. $\downarrow(S) = \{x \in X : \exists\, s \in S\ x \leqslant s\}$.

NB. The partial order $\leqslant$ on A resulting from this structure is the *opposite* of the subset inclusion, i. e. $D \leqslant D'$ iff $D \supseteq D'$.

In fact it follows from the work of Köhler [3] that every *finite* implicative semilattice is isomorphic to one of the above form. Moreover, Köhler obtained a dual description of homomorphisms between finite implicative semilattices in terms of certain partial maps between posets.

We extend this finite duality of Köhler to nuclear algebras. Every subset $S \subseteq X$ of a poset $(X, \leqslant)$ gives rise to a nucleus $\mathbf{j}_S : \mathbf{D}(X, \leqslant) \to \mathbf{D}(X, \leqslant)$ defined by $\mathbf{j}_S(D) = \downarrow(S \cap D)$. Moreover for finite X, every nucleus $\mathbf{j}$ on $\mathbf{D}(X, \leqslant)$ is equal to some such $\mathbf{j}_S$, for a unique subset $S \subseteq X$. We also obtain description of homomorphisms of nuclear algebras in terms of partial maps between the corresponding posets, as in [3].

This in turn makes it possible to give a dual description of nuclear subalgebras, and a dual characterization of situations when a nuclear algebra A is generated by its elements $a_1, ..., a_n \in A$. In particular, we have

Theorem 1. *Given downsets $D_1, ..., D_n$ of a finite poset $(X, \leqslant)$ and a subset $S \subseteq X$, if the nuclear algebra $(\mathbf{D}(X, \leqslant), \mathbf{j}_S)$ is generated by its elements $D_1, ..., D_n$ then for any $x \in X$, either $x \in \max(D_k)$ for some $k \in \{1, ..., n\}$ or $x \in \max(S \cap \downarrow y)$ for some $y \neq x$.*

This then enables us to apply the general construction of the universal model from [2] to our case.

Given a natural number n, we construct a poset $L(n)$, a subset $S(n) \subseteq L(n)$ and downsets $D(n, 1), ..., D(n, n) \in \mathbf{D}(L(n))$ with the following universal property: for any finite poset X and any subset $S \subseteq X$, if the nuclear algebra $(\mathbf{D}(X), \mathbf{j}_S)$ is generated by the downsets $D_1, ..., D_n \in \mathbf{D}(X)$, then there is a unique isomorphism $\varphi : X \to X'$ to a downset $X' \subseteq L(n)$ with $\varphi(S) = X' \cap S(n)$ and $\varphi(D_k) = X' \cap D(n, k)$, $k = 1, ..., n$.

The construction is inductive: we start with $L(n)_0$ empty; having constructed $S(n)_i \subseteq L(n)_i$ and $D(n, 1)_i, ..., D(n, n)_i \in \mathbf{D}(L(n)_i)$, we define $L(n)_{i+1} \supseteq L(n)_i$, $S(n)_{i+1} \supseteq S(n)_i$, $D(n, k)_{i+1} \supseteq D(n, k)_i$, $k = 1, ..., n$, as follows.

$L(n)_{i+1} \setminus L(n)_i$ consists of elements $r_{\alpha,\sigma} \notin S(n)_{i+1}$, one for each antichain α in $L(n)_i$, with $\alpha \nsubseteq L(n)_{i-1}$ if $i > 0$, and each $\sigma \subsetneqq \{k \in \{1, ..., n\} : \alpha \subseteq D(n, k)_i\}$, as well as elements $s_{\alpha,\sigma} \in S(n)_{i+1}$, one for each such pair α, σ that $\sigma \subseteq \{k \in \{1, ..., n\} : \alpha \subseteq D(n, k)_i\}$ and either $\sigma \neq \{k \in \{1, ..., n\} : \alpha \subseteq D(n, k)_i\}$ or $\alpha \nsubseteq S(n)_i$.

We then define $D(n, k)_{i+1} = D(n, k)_i \cup \{r_{\alpha,\sigma} : k \in \sigma\} \cup \{s_{\alpha,\sigma} : k \in \sigma\}$, $k = 1, ..., n$.

Extension of the partial order to $L(n)_{i+1}$ is uniquely determined by the requirements $\max(\downarrow(r_{\alpha,\sigma}) \setminus \{r_{\alpha,\sigma}\}) = \alpha$ and $\max(\downarrow(s_{\alpha,\sigma}) \setminus \{s_{\alpha,\sigma}\}) = \alpha$.

We then have

Theorem 2. *The above construction stops after finite number of steps, i. e. there is an i such that $L(n) = L(n)_i$ with $S(n) = S(n)_i$, $D(n, k) = D(n, k)_i$ has the above universal property.*

On the other hand we have

Theorem 3. *The variety of nuclear algebras has the finite model property.*

These facts enable us to conclude

Corollary. *For each n, the finite nuclear algebra $(\mathbf{D}(L(n)), \mathbf{j}_{S(n)})$ is the free n-generated nuclear algebra. In particular, the variety of nuclear algebras is locally finite.*

References

[1] A. Diego. *Sur les Algèbres de Hilbert.* Translated from the Spanish by Luisa Iturrioz. Collection de Logique Mathématique, Sér. A, Fasc. XXI. Gauthier-Villars, Paris, 1966.

[2] Silvio Ghilardi. Irreducible models and definable embeddings. In *Logic Colloquium '92 (Veszprém, 1992)*, Stud. Logic Lang. Inform., pages 95–113. CSLI Publ., Stanford, CA, 1995.

[3] P. Köhler. Brouwerian semilattices. *Trans. Amer. Math. Soc.*, 268(1):103–126, 1981.