Algebras from a Quasitopos of Rough Sets

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Rough set theory was defined by Pawlak [3] to deal with incomplete information. Since then it has been studied from many directions including algebra and category theory. A summary of previous work on categories of rough sets can be found in [2]. Our work is an amalgamation of the algebraic and category-theoretic approaches. In this work, we introduce the class of contrapositionally complemented pseudo-Boolean algebras and the corresponding logic, emerging from the study of algebras of strong subobjects in a generalized category of rough sets.

Elementary topoi were defined to capture properties of the category of sets. With a similar goal in mind, in [2] we proposed the following natural generalization RSC(\mathcal{E}) of the category RSC of rough sets. RSC has the pairs \((X_1, X_2)\) as objects, where \(X_1, X_2\) are sets and \(X_1 \subseteq X_2\), and the set functions \(f : X_2 \to Y_2\) as arrows with domain \((X_1, X_2)\) and codomain \((Y_1, Y_2)\) such that \(f(X_1) \subseteq Y_1\). By replacing sets with objects of an arbitrary topos \(\mathcal{E}\), we obtain

Definition 1. [2] The category RSC(\mathcal{E}) has the pairs \((A, B)\) as objects, where \(A\) and \(B\) are \(\mathcal{E}\)-objects such that there exists a monic arrow \(m : A \to B\) in \(\mathcal{E}\). \(m\) is said to be a monic corresponding to the object \((A, B)\). The pairs \((f', f)\) are the arrows with domain \((X_1, X_2)\) and codomain \((Y_1, Y_2)\), where \(f' : X_1 \to Y_1\) and \(f : X_2 \to Y_2\) are \(\mathcal{E}\)-arrows such that \(m f' = f m\), and \(m\) and \(m'\) are monics corresponding to the objects \((X_1, X_2)\) and \((Y_1, Y_2)\) in RSC(\mathcal{E}) respectively.

The category RSC(\mathcal{E}) forms a quasitopos [2]. Any quasitopos, just like a topos, has an internal (intuitionistic) logic associated with the strong subobjects of its objects [6]. Let \(\mathcal{M}((U_1, U_2))\) be the set of strong monics of an RSC(\mathcal{E})-object \((U_1, U_2)\). \(\mathcal{M}((U_1, U_2))\) thus forms a pseudo-Boolean algebra. Moreover, the operations on \(\mathcal{M}((U_1, U_2))\) are characterized as follows.

Proposition 1. The operations on \(\mathcal{M}((U_1, U_2))\) obtained by taking the pullbacks of specific characteristic arrows along the RSC(\mathcal{E})-subobject classifier \((\top, \top) : (1, 1) \to (\Omega, \Omega)\) are:

\[ \cap : (f', f) \cap (g', g) = (f' \cap g', f \cap g), \quad \cup : (f', f) \cup (g', g) = (f' \cup g', f \cup g), \quad \neg : \neg(f', f) = (\neg f',\neg f), \quad \neg\neg : (f', f) \to (g', g) = (f' \to g', f \to g), \]

where \((f', f)\) and \((g', g)\) are strong monics with codomain \((U_1, U_2)\), and \(\top : 1 \to \Omega\) is the subobject classifier of the topos \(\mathcal{E}\). The operations on \(f', g'\) \((f, f)\) used above are those of the algebra of subobjects of \(U_1\) \((U_2)\) in the topos \(\mathcal{E}\).

In the context of the algebra of strong subobjects of an RSC-object \((U_1, U_2)\), we had noted in [2] that, since the complementation \(\neg\) is with respect to the object \((U_1, U_2)\), we actually require the concept of relative rough complementation. Iwiński’s rough difference operator [1] is what we use, and we define a new negation \(\sim\) on \(\mathcal{M}((U_1, U_2))\) as:

\[ \sim (f', f) := (\neg f',\neg (m \circ f')), \]

where \((f', f)\) is a strong monic with codomain \((U_1, U_2)\) and \(m : U_1 \to U_2\) is a monic arrow corresponding to \((U_1, U_2)\). We observe that \(A := (\mathcal{M}((U_1, U_2)), (Id_{U_1}, Id_{U_2}), \cap, \cup, \rightarrow, \sim)\) forms a contrapositionally complemented (c.c.) lattice [5], with \(1 := (Id_{U_1}, Id_{U_2})\). In fact, \(A\) satisfies the property \(\sim a = a \to \neg \neg \sim 1\), which is not true in general for an arbitrary c.c. lattice. Moreover, \(\sim\) is neither a semi-negation nor involutive. These observations indicate that \(A\) is an instance of a new algebraic structure, involving two negations \(\sim\) and \(\neg\), and defined as follows.
Definition 2. An abstract algebra \( A := (A, 1, 0, \rightarrow, \cup, \cap, \neg, \sim) \) is said to be a contrapositio-
nally completed pseudo-Boolean algebra (c.c.-pseudo-Boolean algebra) if \((A, 1, 0, \rightarrow, \cup, \cap, \neg)\) forms a pseudo-Boolean algebra and \( \sim a = a \rightarrow (\neg \neg \sim a) \), for all \( a \in A \).

An entire class of c.c.-pseudo-Boolean algebras can be obtained as follows, starting from any pseudo-
Boolean algebra \( \mathcal{H} := (H, 1, 0, \rightarrow, \cup, \cap, \neg) \).

Theorem 2. Let \( \mathcal{H}^2 := \{(a, b) : a \leq b, a, b \in H\} \), and \( u := (u_1, u_2) \in \mathcal{H}^2 \). Consider the set \( A_u := \{(a_1, a_2) \in \mathcal{H}^2 : a_2 \leq u_2 \) and \( a_1 = a_2 \wedge u_3\} \). Define the following operators on \( A_u \):

\[
\begin{align*}
\sqcup & : (a_1, a_2) \sqcup (b_1, b_2) := (a_1 \vee b_1, a_2 \vee b_2), \\
\cap & : (a_1, a_2) \cap (b_1, b_2) := (a_1 \wedge b_1, a_2 \wedge b_2), \\
\neg & : \neg (a_1, a_2) := (u_1 \wedge \neg a_1, u_2 \wedge \neg a_2), \\
\sim & : \sim (a_1, a_2) := (u_1 \wedge \neg a_1, u_2 \wedge \neg a_1), \\
\rightarrow & : (a_1, a_2) \rightarrow (b_1, b_2) := ((a_1 \rightarrow b_1) \wedge u_1, (a_2 \rightarrow b_2) \wedge u_2).
\end{align*}
\]

Then \( A_u := (A_u, u, (0, 0), \rightarrow, \cup, \cap, \neg, \sim) \) is a c.c.-pseudo-Boolean algebra.

We define in the standard way, a c.c.-pseudo-Boolean set lattice. Using the representation theo-
rem for pseudo-Boolean algebras \([5]\), one obtains the following.

Theorem 3 (Representation Theorem). Let \( A := (A, 1, 0, \rightarrow, \cup, \cap, \neg, \sim) \) be a c.c.-pseudo-
Boolean algebra. There exists a monomorphism \( h \) from \( A \) into a c.c.-pseudo-Boolean set lattice.

Note that, as the class of all pseudo-Boolean algebras is equationally definable, the class 
of all c.c.-pseudo-Boolean algebras is also so. Thus we define the logic corresponding to c.c.-
pseudo-Boolean algebras, and call it Intuitionistic logic with minimal negation (ILM).

Various definitions of mappings from one formal system to another can be found in liter-
ature. A detailed study of connections between Classical logic (CL), Intuitionistic logic (IL) 
and Minimal logic (ML) can be found in \([4]\), which has first formally defined the notion of
‘interpretability’ of formulas of one logic into another. In our work, we generalize the notion as 
follows. The mapping \( r : L_1 \rightarrow L_2 \) from formulas in logic \( L_1 \) to formulas in logic \( L_2 \) is called 
an interpretation, if for any formula \( \alpha \in L_1 \), we have \( r \models L_1 \alpha \) if and only if \( \Delta_\alpha \models L_2 r(\alpha) \), where \( \Delta_\alpha \) is a finite set of formulas in \( L_2 \) corresponding to \( \alpha \). \( r \) is an embedding, if it is the inclusion 
map and \( \Delta_\alpha = \emptyset \) for any \( \alpha \in L_1 \). IL can clearly be embedded into ILM. Furthermore, we have 

Theorem 4. There exists an interpretation from ILM into IL.

The proof is similar to the one used to show connections between constructive logic with strong 
negation \([5]\, Chapter XII) and IL.

We may also compare ILM and ML. Since ML corresponds to the class of c.c lattices \([5]\) 
and any c.c.-pseudo-Boolean algebra is a c.c. lattice, ML can be embedded inside ILM. Using 
Theorem 4 and an interpretation of IL into ML \([4, \text{ Theorem B}]\), we have 

Corollary 5. There exists an interpretation from ILM into ML.

References