Algebras from a Quasitopos of Rough Sets

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Rough set theory was defined by Pawlak [3] to deal with incomplete information. Since then it has been studied from many directions including algebra and category theory. A summary of previous work on categories of rough sets can be found in [2]. Our work is an amalgamation of the algebraic and category-theoretic approaches. In this work, we introduce the class of *contrapositionally complemented pseudo-Boolean algebras* and the corresponding logic, emerging from the study of algebras of strong subobjects in a generalized category of rough sets.

Elementary topoi were defined to capture properties of the category of sets. With a similar goal in mind, in [2] we proposed the following natural generalization $RSC(\mathscr{C})$ of the category RSC of rough sets. RSC has the pairs (X_1, X_2) as objects, where X_1, X_2 are sets and $X_1 \subseteq X_2$, and the set functions $f : X_2 \to Y_2$ as arrows with domain (X_1, X_2) and codomain (Y_1, Y_2) such that $f(X_1) \subseteq Y_1$. By replacing sets with objects of an arbitrary topos $\mathscr{C}$, we obtain

Definition 1. *[2] The category $RSC(\mathscr{C})$ has the pairs (A, B) as objects, where A and B are $\mathscr{C}$-objects such that there exists a monic arrow $m : A \to B$ in $\mathscr{C}$. m is said to be a* monic corresponding to the object (A, B). *The pairs (f', f) are the arrows with domain (X_1, X_2) and codomain (Y_1, Y_2), where $f' : X_1 \to Y_1$ and $f : X_2 \to Y_2$ are $\mathscr{C}$-arrows such that $m'f' = fm$, and m and m' are monics corresponding to the objects (X_1, X_2) and (Y_1, Y_2) in $RSC(\mathscr{C})$ respectively.*

The category $RSC(\mathscr{C})$ forms a quasitopos [2]. Any quasitopos, just like a topos, has an internal (intuitionistic) logic associated with the strong subobjects of its objects [6]. Let $\mathcal{M}((U_1, U_2))$ be the set of strong monics of an $RSC(\mathscr{C})$-object (U_1, U_2). $\mathcal{M}((U_1, U_2))$ thus forms a pseudo-Boolean algebra. Moreover, the operations on $\mathcal{M}((U_1, U_2))$ are characterized as follows.

Proposition 1. *The operations on $\mathcal{M}((U_1, U_2))$ obtained by taking the pullbacks of specific characteristic arrows along the $RSC(\mathscr{C})$-subobject classifier $(\top, \top) : (1, 1) \to (\Omega, \Omega)$ are:*

$\cap:\ (f', f) \cap (g', g) = (f' \cap g', f \cap g),\quad \cup:\ (f', f) \cup (g', g) = (f' \cup g', f \cup g),$

$\neg:\ \neg(f', f) = (\neg f', \neg f),\quad \to:\ (f', f) \to (g', g) = (f' \to g', f \to g),$

where (f', f) and (g', g) are strong monics with codomain (U_1, U_2), and $\top : 1 \to \Omega$ is the subobject classifier of the topos $\mathscr{C}$. The operations on f', g' (f, g) used above are those of the algebra of subobjects of U_1 (U_2) in the topos $\mathscr{C}$.

In the context of the algebra of strong subobjects of an RSC-object (U_1, U_2), we had noted in [2] that, since the complementation $\neg$ is with respect to the object (U_1, U_2), we actually require the concept of *relative* rough complementation. Iwiński's *rough difference* operator [1] is what we use, and we define a new negation $\sim$ on $\mathcal{M}((U_1, U_2))$ as:

$$\sim (f', f) := (\neg f', \neg(m \circ f')),$$

where (f', f) is a strong monic with codomain (U_1, U_2) and $m : U_1 \to U_2$ is a monic arrow corresponding to (U_1, U_2). We observe that $\mathcal{A} := (\mathcal{M}((U_1, U_2)), (Id_{U_1}, Id_{U_2}), \cap, \cup, \to, \sim)$ forms a contrapositionally complemented (*c.c.*) lattice [5], with $1 := (Id_{U_1}, Id_{U_2})$. In fact, $\mathcal{A}$ satisfies the property $\sim a = a \to \neg\neg \sim 1$, which is not true in general for an arbitrary *c.c.* lattice. Moreover, $\sim$ is neither a semi-negation nor involutive. These observations indicate that $\mathcal{A}$ is an instance of a new algebraic structure, involving two negations $\sim$ and $\neg$, and defined as follows.

Definition 2. *An abstract algebra* $\mathcal{A} := (A, 1, 0, \rightarrow, \cup, \cap, \neg, \sim)$ *is said to be a* contrapositionally complemented pseudo-Boolean algebra *(c.c.-pseudo-Boolean algebra) if* $(A, 1, 0, \rightarrow, \cup, \cap, \neg)$ *forms a pseudo-Boolean algebra and* $\sim a = a \rightarrow (\neg\neg \sim 1)$*, for all* $a \in A$*.*

An entire class of *c.c.*-pseudo-Boolean algebras can be obtained as follows, starting from any pseudo-Boolean algebra $\mathcal{H} := (H, 1, 0, \rightarrow, \cup, \cap, \neg)$.

Theorem 2. *Let* $\mathcal{H}^{[2]} := \{(a, b) : \ a \leq b, \ a, b \in H\}$*, and* $u := (u_1, u_2) \in \mathcal{H}^{[2]}$*. Consider the set* $A_u := \{(a_1, a_2) \in \mathcal{H}^{[2]} \ : a_2 \leq u_2 \text{ and } \ a_1 = a_2 \wedge u_1\}$*. Define the following operators on* A_u*:*

$\sqcup :\ (a_1, a_2) \sqcup (b_1, b_2) := (a_1 \vee b_1, a_2 \vee b_2), \qquad \sqcap :\ (a_1, a_2) \sqcap (b_1, b_2) := (a_1 \wedge b_1, a_2 \wedge b_2),$

$\neg :\ \neg(a_1, a_2) := (u_1 \wedge \neg a_1, u_2 \wedge \neg a_2), \qquad \sim:\ \sim (a_1, a_2) := (u_1 \wedge \neg a_1, u_2 \wedge \neg a_1),$

$\rightarrow:\ (a_1, a_2) \rightarrow (b_1, b_2) := ((a_1 \rightarrow b_1) \wedge u_1, (a_2 \rightarrow b_2) \wedge u_2).$

Then $\mathcal{A}_u := (A_u, u, (0, 0), \rightarrow, \sqcup, \sqcap, \neg, \sim)$ *is a c.c.-pseudo-Boolean algebra.*

We define in the standard way, a *c.c.*-pseudo-Boolean set lattice. Using the representation theorem for pseudo-Boolean algebras [5], one obtains the following.

Theorem 3 (Representation Theorem). *Let* $\mathcal{A} := (A, 1, 0, \rightarrow, \cup, \cap, \neg, \sim)$ *be a c.c.-pseudo-Boolean algebra. There exists a monomorphism* h *from* $\mathcal{A}$ *into a c.c.-pseudo-Boolean set lattice.*

Note that, as the class of all pseudo-Boolean algebras is equationally definable, the class of all *c.c.*-pseudo-Boolean algebras is also so. Thus we define the logic corresponding to *c.c.*-pseudo-Boolean algebras, and call it *Intuitionistic logic with minimal negation* (ILM).

Various definitions of mappings from one formal system to another can be found in literature. A detailed study of connections between Classical logic (CL), Intuitionistic logic (IL) and Minimal logic (ML) can be found in [4], which has first formally defined the notion of 'interpretability' of formulas of one logic into another. In our work, we generalize the notion as follows. The mapping $r : L_1 \rightarrow L_2$ from formulas in logic L_1 to formulas in logic L_2 is called an *interpretation*, if for any formula $\alpha \in L_1$, we have $\vdash_{L_1} \alpha$ if and only if $\Delta_\alpha \vdash_{L_2} r(\alpha)$, where Δ_α is a finite set of formulas in L_2 corresponding to α. r is an embedding, if it is the inclusion map and $\Delta_\alpha = \emptyset$ for any α in L_1. IL can clearly be embedded into ILM. Furthermore, we have

Theorem 4. *There exists an interpretation from* ILM *into* IL*.*

The proof is similar to the one used to show connections between constructive logic with strong negation [5, Chapter XII] and IL.

We may also compare ILM and ML. Since ML corresponds to the class of *c.c* lattices [5] and any *c.c.*-pseudo-Boolean algebra is a *c.c.* lattice, ML can be embedded inside ILM. Using Theorem 4 and an interpretation of IL into ML [4, Theorem B], we have

Corollary 5. *There exists an interpretation from* ILM *into* ML*.*

References

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