A new characterization of a class $\mathrm{HSP_U}(\mathcal{K})$

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We recall that a class of algebras $\mathcal{K}$ has *f*inite embeddability property, briefly (FEP), if any finite partial sublagebra of any member is embeddable into some finite one from $\mathcal{K}$. This property was generalized in [1] by the following way.

Definition 1. *Let $\mathbf{A} = (A, \mathrm{F})$ be an algebra and $X \subseteq A$. A* partial subalgebra *is a pair $\mathbf{A}|_X = (X, \mathrm{F})$, where for any $f \in F_n$ and all $x_1, \dots, x_n \in X$, $f^{\mathbf{A}|_X}(x_1, \dots, x_n)$ is defined if and only if $f^{\mathbf{A}}(x_1, \dots, x_n) \in X$. We put then*

$$f^{\mathbf{A}|_X}(x_1, \dots, x_n) := f^{\mathbf{A}}(x_1, \dots, x_n).$$

Definition 2. *An algebra $\mathbf{A} = (A; \mathrm{F})$ satisfies* the general finite embeddability *(*finite embeddability property*)* property for the class $\mathcal{K}$ *of algebras of the same type* F *if for any finite subset $X \subseteq A$, there exist a (finite) algebra $\mathbf{B} \in \mathcal{K}$ and an embedding $\rho : \mathbf{A}|_X \hookrightarrow \mathbf{B}$, i.e., an injective mapping $\rho : X \to B$ satisfying the property $\rho(f^{\mathbf{A}|_X}(x_1, \dots, x_n)) = f^{\mathbf{B}}(\rho(x_1), \dots, \rho(x_n))$ if $x_1, \dots, x_n \in X$, $f \in \mathrm{F}_n$ and $f^{\mathbf{A}|_X}(x_1, \dots, x_n)$ is defined.*

The most important idea of this generalization is described in the following theorems. Both theorems are direct consequences of well known model theory facts (see [4]) or alternatively the direct proofs are published in [1].

Theorem 1. *Let $\mathbf{A} = (A; \mathrm{F})$ be an algebra and let $\mathcal{K}$ be a class of algebras of the type F. If $\mathbf{A}$ satisfies the general finite embeddability property for $\mathcal{K}$ then $\mathbf{A} \in \mathrm{ISP_U}(\mathcal{K})$.*

Theorem 2. *Let $\mathbf{A} = (A; \mathrm{F})$ be an algebra such that F is finite and let $\mathcal{K}$ be a class of algebras of the type F. If $\mathbf{A} \in \mathrm{ISP_U}(\mathcal{K})$ then $\mathbf{A}$ satisfies the general finite embeddability property for $\mathcal{K}$.*

The great applicability of these concepts in logic theory is obvious. Examples of results obtained using by some type of partial embeddability are contained for example in [1, 2, 5, 6, 7] etc.

In our talk we present analogous property (a finite covering property of an algebra $\mathbf{A}$ by a class $\mathcal{K}$) and then we show that this property is equivalent to $\mathbf{A} \in \mathrm{HSP_U}(\mathcal{K})$.

In next we denote the set of all terms of type F over the set A by $\mathbf{T}_{\mathrm{F}}(A)$. Let $t \in \mathbf{T}_{\mathrm{F}}(A)$ be a term then we denote by $|t|$ the set of all variables (members of A) used in the term t. For any set $T \subseteq \mathbf{T}_{\mathrm{F}}(A)$ we denote

$$|T| = \bigcup_{t \in t} |t|.$$

Now we are ready to define the finite covering property.

Definition 3. *Let $\mathbf{A} = (A, \mathrm{F})$ be an algebra and $\mathcal{K}$ be a class of algebras of the type F. We say that the class $\mathcal{K}$* finitely partially covers the algebra $\mathbf{A}$ *(or $\mathbf{A}$ satisfies* the finite covering property for the class $\mathcal{K}$*) if for every finite set of terms $T \subseteq \mathbf{T}_{\mathrm{F}}(A)$ there exist an algebra $\mathbf{B} \in \mathcal{K}$, a mapping $f \colon B \to A$ and a set $Y \subseteq B$ such that*

i) $f|_Y \colon Y \to |T|$ is a bijection, *ii) if* $t(a_1, \ldots, a_n) \in T$ *and* $y_1, \ldots, y_n \in Y$ *are such that* $fy_i = a_i$ *then*

$$ft^{\mathcal{B}}(y_1, \ldots, y_n) = t^{\mathcal{A}}(a_1, \ldots, a_n).$$

The following theorem is the most important result of our talk.

Theorem 3. *Let* $\mathbf{A}$ *satisfy the finite covering property for the class* $\mathcal{K}$ *if and only if* $\mathbf{A} \in \mathrm{HSP_U}(\mathcal{K})$.

This theorem has several natural consequences, for example, using well known Bjarni Jónsson Lemma (see [3]) we obtain:

Theorem 4. *Let* $\mathcal{K}$ *be a generating class of congruence distributive variety* $\mathcal{V}$ *then any subdirectly irreducible member of* $\mathcal{V}$ *is finitely coverable by* $\mathcal{K}$*.*

References

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