# On interpolation in $NEXT(\mathbf{KTB})$ and $NEXT(\mathbf{KB})$

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We study two modal logics: the Brouwer logic $\mathbf{KTB} := \mathbf{K} \oplus T \oplus B$ and its interesting sub-logic – the logic $\mathbf{KB} := \mathbf{K} \oplus B$, where:

$$T := \Box p \to p$$
$$B := p \to \Box \Diamond p$$

The logic **KTB** (logic **KB**) is complete with respect to the class of reflexive and symmetric Kripke frames (symmetric Kripke frames).
We shall study n-branching Brouwerian modal logics $\mathbf{KTB.Alt(n)} := \mathbf{KTB} \oplus alt_n$ as well as $\mathbf{KB.Alt(n)} := \mathbf{KB} \oplus alt_n$ where

$$alt_n := \Box p_1 \vee \Box(p_1 \to p_2) \vee ... \vee \Box((p_1 \wedge ... \wedge p_n) \to p_{n+1}).$$

For $n = 3$ the above axiom involves linearity of the appropriate reflexive frames – they are chains of reflexive points. Chains of (possibly) irreflexive points characterize logic $\mathbf{KB.Alt(2)}$.

**Definition 1.** *A logic $L$ has the Craig interpolation property (CIP) if for every implication $\alpha \to \beta$ in $L$, there exists a formula $\gamma$ (interpolant for $\alpha \to \beta$ in $L$) such that*

$$\alpha \to \gamma \in L \ \ and \ \ \gamma \to \beta \in L \quad and \quad Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta).$$

The symbol $Var(\alpha)$ means the set of all propositional variables of the formula $\alpha$.
The weaker notion of interpolation for deducibility is defined as follows:

**Definition 2.** *A logic $L$ has interpolation for deducibility (IPD) if for any $\alpha$ and $\beta$ the condition $\alpha \vdash_L \beta$ implies that there exists a formula $\gamma$ such that*

$$\alpha \vdash_L \gamma \ \ and \ \ \gamma \vdash_L \beta, \ \ and \ \ Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta).$$

It is a logical folklore that (CIP) together with (MP) and deduction theorem implies (IPD).

It is known that **K**, **T**, **K4** and **S4** have (CIP), see Gabbay [3]. Also the logics from $NEXT(\mathbf{S4})$ are well characterized as regards interpolation (see [5], also [1], p.462-463). It is also known that **S5** has (CIP). The last fact can be proven by applying a very general method of construction of inseparable tableaux (see i.e. [1], p. 446-449). The same method can be applied in the case of **KTB** and **KB**. Then we get that the logics **KTB** and **KB** have (CIP).

The following facts were proven in [4]:

**Theorem 1.** *The logic $\mathbf{KTB.Alt(3)}$ does not have (CIP).*

**Theorem 2.** *There are only two tabular logics from $NEXT(\mathbf{KTB.Alt(3)})$ having (IPD). They are the trivial logic $L(\circ)$ and the logic determined by two element cluster $L(\circ - - \circ)$.*

In [4] the following conjectures are placed:

**Conjecture 1.** *The logic determined by a reflexive and symmetric Kripke frame having the structure of a Boolean cube has (IDP).*

**Conjecture 2.** *The logic determined by a reflexive and symmetric Kripke frame having the structure of* $2^n$*-element Boolean cube,* $n \geq 3$*, has (IDP).*

In our talk we disprove these conjectures and prove others negative results on interpolation in $NEXT(\mathbf{KTB}.\mathbf{Alt}(\mathbf{n}))$ for $n \geq 3$. We also provide a similar research for the logics from $NEXT(\mathbf{KB}.\mathbf{Alt}(\mathbf{n}))$. First result, a similar to Theorem 1 is the following:

**Theorem 3.** *The logic* $\mathbf{KB}.\mathbf{Alt}(\mathbf{2})$ *does not have (CIP).*

Second, in contrast to logics from $NEXT(\mathbf{KTB}.\mathbf{Alt}(\mathbf{3}))$ we prove:

**Theorem 4.** *There are infinitely many tabular logics from* $NEXT(\mathbf{KB}.\mathbf{Alt}(\mathbf{2}))$ *having (IPD).*

# References

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