$\aleph_1$, ω_1, and the modal μ-calculus

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The modal μ-calculus $\mathbf{L}_\mu$, see [4], enriches the syntax of (poly)modal logic $\mathbf{K}$ with least and greatest fixed-point constructors μ and ν. In a Kripke model $\mathcal{M}$, the formula $\mu_x.\phi$ (resp., $\nu_x.\phi$) denotes the least (resp., the greatest) fixed-point of the function $\phi_\mathcal{M}$ (of the variable x) obtained by evaluating ϕ in $\mathcal{M}$ under the additional condition that x is interpreted as a given subset of worlds. It is required that every occurrence of x is positive in ϕ, so $\phi_\mathcal{M}$ is monotone and the least fixed-point exists by the Tarski-Knaster theorem.

A formula $\phi(x)$ is said to be continuous if, for every model $\mathcal{M}$, the function $\phi_\mathcal{M}$ is continuous, in the usual sense. The continuous fragment $\mathcal{C}_0(X)$ of the modal μ-calculus is the set of formulas generated by the following syntax:

$$\phi := x \mid \psi \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \langle a \rangle \phi \mid \mu_z.\chi\,,$$

where $x \in X$, $\psi \in \mathbf{L}_\mu$ is a μ-calculus formula not containing any variable $x \in X$, and $\chi \in \mathcal{C}_0(X \cup \{ z \})$. Fontaine [3] proved that a formula $\phi \in \mathbf{L}_\mu$ is continuous in x if and only if it is equivalent to a formula in $\mathcal{C}_0(x)$; she also proved that it is decidable whether a formula of the modal μ-calculus is continuous. We add to the above grammar one more production and study the fragment $\mathcal{C}_1(X)$ of $\mathbf{L}_\mu$ defined as follows:

$$\phi := x \mid \psi \mid \top \mid \bot \mid \phi \wedge \phi \ \mid \phi \vee \phi \mid \langle a \rangle \phi \mid \mu_z.\chi \mid \nu_z.\chi\,,$$

with the same constraints as above but w.r.t $\mathcal{C}_1(X \cup \{ z \})$.

Definition 1. Let κ be a regular cardinal. A set $\mathcal{I} \subseteq P(X)$ is *κ-directed* if every subset of $\mathcal{I}$ of cardinality smaller than κ has an upper bound in $\mathcal{I}$. A function $f : P(X) \to P(X)$ is *κ-continuous* if it preserves unions of κ-directed sets.

Notice that, if $\kappa = \aleph_0$, then κ-continuity is the standard notion of continuity. The following proposition is an immediate consequence of the fact that $\aleph_1$-continuous functions are closed under parametrized least and greatest fixed-points, see [5, 6].

Proposition 2. *Every formula in $\phi(x) \in \mathcal{C}_1(x)$ is $\aleph_1$-continuous.*

The folllowing theorem is a sort of converse to the previous statement.

Theorem 3. *For each formula $\phi(x) \in \mathbf{L}_\mu$ we can construct a formula $\psi(x) \in \mathcal{C}_1(x)$ such that $\phi(x)$ is κ-continuous for some regular cardinal κ if and only if $\phi(x)$ is equivalent to $\psi(x)$.*

The consequences of this theorem are twofold.

Corollary 4. *It is decidable whether a formula $\phi(x)$ is κ-continuous for some regular cardinal κ.*

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Corollary 5. *If a formula is κ-continuous for some regular cardinal κ, then it is κ-continuous for some $\kappa \in \{ \aleph_0, \aleph_1 \}$.*

That is, there are no other relevant fragments of the modal μ-calculus, apart from $\mathcal{C}_0$ and $\mathcal{C}_1$, that are determined from some continuity condition.

Let us recall that, for a monotone function $f : P(X) \to P(X)$, we can define the *approximants to the least fixed-point of* f as follows: $f^{\alpha+1}(\emptyset) = f(f^{\alpha}(\emptyset))$ and $f^{\beta}(\emptyset) = \bigcup_{\alpha<\beta} f^{\alpha}(\emptyset)$ (so $f^0(\emptyset) = \emptyset$). If $f^{\alpha+1}(\emptyset) = f^{\alpha}(\emptyset)$, then $f^{\alpha}(\emptyset)$ is the least fixed-point of f.

Definition 6. We say that and ordinal α is the *closure ordinal* of $\phi(x) \in \mathbf{L}_\mu$ if, for every model $\mathcal{M}$, $\phi^{\alpha}_{\mathcal{M}}(\emptyset)$ is the least fixed-point of $\phi_{\mathcal{M}}$, and moreover there exists a model $\mathcal{M}$ for which $\phi^{\beta}(\emptyset)$ is not the least fixed-point of $\phi_{\mathcal{M}}$, for every $\beta < \alpha$.

Of course, not every formula $\phi(x) \in \mathbf{L}_\mu$ has a closure ordinal. For example $[\ \ ]x$ has no closure ordinal, while ω_0 is the closure ordinal of $[\ \ ]\bot \vee \langle\ \ \rangle x$. Czarnecki [2] proved that every ordinal $\alpha < \omega_0^2$ is the closure ordinal of a formula $\phi \in \mathbf{L}_\mu$. Afshari and Leigh [1] proved that if a formula $\phi(x) \in \mathbf{L}_\mu$ does not contain greatest fixed-points and has a closure ordinal α, then $\alpha < \omega_0^2$. Considering that every ordinal below ω_0^2 can be written as a polynomial in the inderterminates $1, \omega_0$, our next theorem can be used to recover Czarnecki's result:

Theorem 7. *Closure ordinals of formulas of the modal μ-calculus are closed under ordinal sum.*

Since a formula $\phi(x)$ in the syntactic fragment $\mathcal{C}_1(x)$ is $\aleph_1$-continuous, the maps $\phi_{\mathcal{M}}$ converge to their least fixed-point in at most ω_1 steps, where ω_1 is the least uncountable ordinal (considering cardinals as specific ordinals, we have $\omega_1 = \aleph_1$). In particular, every formula in this fragment has a closure ordinal with ω_1 as an upper bound. We prove that ω_1 is indeed a closure ordinal:

Theorem 8. *ω_1 is the closure ordinal of the formula $\phi(x) := \nu_z.(\langle v\rangle x \wedge \langle h\rangle z) \vee [v]\bot$.*

Extending Thomason's coding to the full modal μ-calculus, it is also possible to construct a monomodal formula in $\mathbf{L}_\mu$ whose only free variable is x, with ω_1 as closure ordinal. Consequently, we extend Czarnecki's result by showing that polynomials in the inderterminates $1, \omega_0, \omega_1$ denote closure ordinals.

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