Varieties of De Morgan Monoids II: Covers of Atoms

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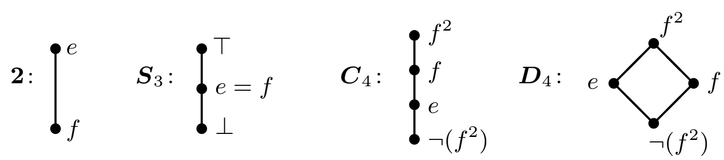
This is the second half of a two-part talk on the lattice Λ_{DMM} of subvarieties of the variety DMM of all De Morgan monoids. The investigation is motivated by an anti-isomorphism between Λ_{DMM} and the lattice of axiomatic extensions of the relevance logic $\mathbf{R^t}$ of [1]. Recall that a De Morgan monoid $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, e\rangle$ is the expansion of a commutative monoid $\langle A; \cdot, e\rangle$ by a residuated distributive lattice order and a compatible antitone involution $\neg$, where $a \leqslant a^2$ for all elements a, and that $f := \neg e$.

The first talk established that the atoms of Λ_{DMM} (i.e., the minimal varieties of De Morgan monoids) are just the four varieties generated, respectively, by the De Morgan monoids depicted below. They include the two-element Boolean algebra $\mathbf{2}$, and the three-element Sugihara monoid $\boldsymbol{S}_3$. In the present talk, we aim to say as much as possible about the covers of these four atoms in Λ_{DMM}, since these define the 'pre-maximal' consistent axiomatic extensions of $\mathbf{R^t}$.



In Λ_{DMM}, a cover K of one of the atoms ($\mathbb{V}(\boldsymbol{A})$, say) will be called *interesting* if K is not the varietal join of $\mathbb{V}(\boldsymbol{A})$ and one of the other three minimal varieties. We can show:

Theorem 1. *(i)* $\mathbb{V}(\mathbf{2})$ *has no interesting cover within* DMM.

(ii) The only interesting cover of $\mathbb{V}(\boldsymbol{S}_3)$ *within* DMM *is the variety* $\mathbb{V}(\boldsymbol{S}_5)$ *generated by the five-element (totally ordered) Sugihara monoid.*

(iii) Every interesting cover of $\mathbb{V}(\boldsymbol{D}_4)$ *within* DMM *has the form* $\mathbb{V}(\boldsymbol{A})$ *for some simple* 1*–generated De Morgan monoid* $\boldsymbol{A}$*, where* $\boldsymbol{D}_4$ *embeds into* $\boldsymbol{A}$ *but is not isomorphic to* $\boldsymbol{A}$.

The situation with $\mathbb{V}(\boldsymbol{C}_4)$ is more complex, as can be guessed from the following result of Slaney [2]: if $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism from a finitely subdirectly irreducible De Morgan monoid into a nontrivial 0–generated De Morgan monoid, then h is an isomorphism or $\boldsymbol{B} \cong \boldsymbol{C}_4$. This motivates study of the class W of all De Morgan monoids that map homomorphically onto $\boldsymbol{C}_4$ or are trivial, as well as its subclass N, consisting of De Morgan monoids that have $\boldsymbol{C}_4$ as a retract or are trivial. It can be shown that W and N are quasivarieties, but neither is a variety.

Theorem 2. W *has a largest subvariety, denoted here by* U*. Also,* N *has a largest subvariety, denoted here by* M*. The varieties* U *and* M *are finitely axiomatized, and* M *consists of the De Morgan monoids in* U *that satisfy* $e \leqslant f$.

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Theorem 3. *If* K *is an interesting cover of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* DMM, *then exactly one of the following holds.*

(i) $\mathsf{K} \subseteq \mathsf{M}$.

(ii) $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$ *for some finite* 0*–generated subdirectly irreducible De Morgan monoid* $\boldsymbol{A} \in \mathsf{U} \setminus \mathsf{M}$.

(iii) $\mathsf{K} = \mathbb{V}(\boldsymbol{A})$ *for some simple* 1*–generated De Morgan monoid* $\boldsymbol{A}$, *such that* $\boldsymbol{C}_4$ *embeds into* $\boldsymbol{A}$ *but is not isomorphic to* $\boldsymbol{A}$.

As $\mathbb{V}(\boldsymbol{C}_4)$ is the only minimal subvariety of U, all covers of $\mathbb{V}(\boldsymbol{C}_4)$ within U are interesting (i.e., they are not joins of atoms in Λ_{DMM}). Only four De Morgan monoids $\boldsymbol{A}$ satisfy the demand in Theorem 3(ii); they are depicted in Slaney [2], where they are labeled $\boldsymbol{C}_5, \boldsymbol{C}_6, \boldsymbol{C}_7, \boldsymbol{C}_8$. Infinitely many covers of $\mathbb{V}(\boldsymbol{C}_4)$ exemplify Theorem 3(iii). Not all of them are finitely generated varieties, and it appears to be difficult to classify them structurally.

Here, however, we are able to describe completely the covers of $\mathbb{V}(\boldsymbol{C}_4)$ within M, i.e., the witnesses of Theorem 3(i). In particular:

Theorem 4. *There are exactly six covers of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* M. *Consequently, there are just ten covers of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* U. *All ten of these covers are finitely generated varieties.*

A *Dunn monoid* is a distributive commutative residuated lattice, satisfying $x \leqslant x^2$, so De Morgan monoids are just Dunn monoids with a compatible involution. Slaney [3] discusses ways of constructing De Morgan monoids $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ from Dunn monoids $\boldsymbol{B}$, where $\boldsymbol{B}$ is a subalgebra of the Dunn monoid reduct of $\mathrm{S}^{\leqslant}(\boldsymbol{B})$. We refer to these methods as *skew reflection* constructions. Each construction first creates a copy b' of every element b of $\boldsymbol{B}$ and orders the new elements so that $b' \leqslant c'$ iff $c \leqslant b$. A new upper bound 1 and lower bound 0 for all of these elements is introduced, and $b' \cdot c'$ is defined to be 1 for all $b, c \in B$. No element of the form b' is a lower bound of an element of $\boldsymbol{B}$, but certain elements of $\boldsymbol{B}$ may be lower bounds of new elements b' (thus expanding the order relation $\leqslant$ on the superstructure $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ of $\boldsymbol{B}$), subject to certain axioms. The axioms ensure that $\mathrm{S}^{\leqslant}(\boldsymbol{B})$ really is a De Morgan monoid.

We can prove that a De Morgan monoid belongs to U iff it is a subdirect product of skew reflections of Dunn monoids, where the bottom element 0 is meet-irreducible in every subdirect factor. This limits the choices of algebras $\boldsymbol{A}$ such that $\mathbb{V}(\boldsymbol{A})$ generates a cover of $\mathbb{V}(\boldsymbol{C}_4)$ within M. It forces $\boldsymbol{A}$ to be finite, and leads eventually to the proof of Theorem 4.

There are additional motivations for study of M, which come from considerations of structural completeness. The minimal varieties of De Morgan monoids are structurally complete, as are the well-understood varieties of odd Sugihara monoids. We have shown that all remaining structurally complete subvarieties of DMM lie within M, though not all subvarieties of M are structurally complete. The following result is therefore of interest:

Theorem 5. *Every cover of* $\mathbb{V}(\boldsymbol{C}_4)$ *within* M *has no proper subquasivariety other than* $\mathbb{V}(\boldsymbol{C}_4)$, *and is thus (hereditarily) structurally complete.*

References

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