Varieties of De Morgan Monoids I: Minimality and Irreducible Algebras

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A *De Morgan monoid* $\boldsymbol{A} = \langle A; \cdot, \wedge, \vee, \neg, e\rangle$ comprises a distributive lattice $\langle A; \wedge, \vee\rangle$, a commutative monoid $\langle A; \cdot, e\rangle$ satisfying $x \leqslant x^2 := x \cdot x$, and a function $\neg\colon A \to A$, called an *involution*, such that $\boldsymbol{A}$ satisfies $\neg\neg x = x$ and $x \cdot y \leqslant z \iff x \cdot \neg z \leqslant \neg y$. (The derived operations $x \to y := \neg(x \cdot \neg y)$ and $f := \neg e$ turn $\boldsymbol{A}$ into an involutive residuated lattice in the sense of [3].)

The class DMM of all De Morgan monoids is a variety that algebraizes the relevance logic $\mathbf{R^t}$ of [1]. Its lattice of subvarieties Λ_{DMM} is dually isomorphic to the lattice of axiomatic extensions of $\mathbf{R^t}$. A *Sugihara monoid* is a De Morgan monoid that is idempotent, i.e., it satisfies $x^2 = x$. Sugihara monoids are subdirect products of chains. They are locally finite and well-understood (see Dunn's contributions to [1]).

In contrast, relatively little is known about the structure of (i) arbitrary De Morgan monoids and (ii) the lattice Λ_{DMM}. This situation is lamented in [8, p. 263] and [2, Sec. 3.5], which predate many recent papers on residuated lattices. But the latter have concentrated mainly on varieties incomparable with DMM (e.g., Heyting and MV-algebras), larger than DMM (e.g., full Lambek algebras) or smaller (e.g., Sugihara monoids). On the positive side, Slaney [5, 6] showed that the free 0–generated De Morgan monoid is finite, and that there are only seven non-isomorphic subdirectly irreducible 0–generated De Morgan monoids. No finiteness result of this kind holds in the 1–generated case, however. This talk and its sequel report on an attempt to enlarge our knowledge of DMM and its subvariety lattice.

Like any commutative residuated lattice, a De Morgan monoid $\boldsymbol{A}$ is finitely subdirectly irreducible iff its neutral element e is join-irreducible. In this case, however, the extra features of De Morgan monoids imply additional properties, e.g., $\boldsymbol{A}$ consists only of upper bounds of e and lower bounds of f, i.e., $A = [e) \cup (f]$. To this description, we can add a new result:

Theorem 1. *Every finitely subdirectly irreducible De Morgan monoid $\boldsymbol{A}$ consists of an interval subalgebra $[\neg a, a]$ and two chains of idempotent elements, $(\neg a]$ and $[a)$, where a is e or f^2.*

In the former case, $[\neg a, a]$ has at most two elements, and $\boldsymbol{A}$ is a Sugihara monoid. The case $a = f^2 \neq e$ is more challenging, as it involves non-idempotent elements and an order that need not be linear. In both cases, e and f belong to the interval $[\neg a, a]$.

To describe the atoms of Λ_{DMM}, we need to refer to the De Morgan monoids depicted below. (If b is the least element of a De Morgan monoid, then $a \cdot b = b$ for all elements a.) Note that $\mathbf{2}$ is a Boolean algebra, and $\boldsymbol{S}_3$ is a Sugihara monoid. In what follows, $\mathbb{V}(\boldsymbol{A})$ [resp. $\mathbb{Q}(\boldsymbol{A})$] denotes the smallest variety [resp. quasivariety] containing an algebra $\boldsymbol{A}$.

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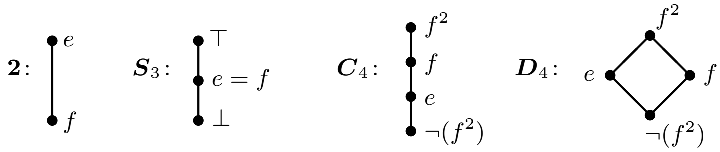

Lemma 2. *Up to isomorphism,* $\mathbf{2}$, $\boldsymbol{C}_4$ *and* $\boldsymbol{D}_4$ *are the only simple* 0*–generated De Morgan monoids.*

Theorem 3. *The distinct classes* $\mathbb{V}(\mathbf{2})$, $\mathbb{V}(\boldsymbol{S}_3)$, $\mathbb{V}(\boldsymbol{C}_4)$ *and* $\mathbb{V}(\boldsymbol{D}_4)$ *are precisely the minimal varieties of De Morgan monoids.*

Lemma 2 is implicit in Slaney's identification of the 0–generated subdirectly irreducible De Morgan monoids, but it is easier to prove it directly. Theorem 3 (which uses Lemma 2) does not seem to have been stated explicitly in the relevance logic literature.

It can also be shown that a subquasivariety of DMM is minimal (i.e., it contains no nontrivial proper subquasivariety) iff it is $\mathbb{V}(\boldsymbol{S}_3)$ or $\mathbb{Q}(\boldsymbol{A})$ for some nontrivial 0–generated De Morgan monoid $\boldsymbol{A}$. Combining this observation with Slaney's description of the free 0–generated De Morgan monoid in [5], we obtain:

Theorem 4. *The variety of De Morgan monoids has just* 68 *minimal subquasivarieties.*

For philosophical reasons, the relevance logic literature also emphasizes a system called $\mathbf{R}$, which differs from $\mathbf{R^t}$ in that it lacks the so-called Ackermann truth constant $\mathbf{t}$ (corresponding to the neutral element e of a De Morgan monoid). The logic $\mathbf{R}$ is algebraized by the variety RA of *relevant algebras*. Świrydowicz [7] showed that the subvariety lattice of RA has a unique atom, with just three covers. We remark that this result can be derived more easily from Theorem 3 and the following finding of Slaney [6]: if $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a homomorphism from a finitely subdirectly irreducible De Morgan monoid into a nontrivial 0–generated De Morgan monoid, then h is an isomorphism or $\boldsymbol{B} \cong \boldsymbol{C}_4$.

Świrydowicz's theorem has been applied recently to show that no consistent axiomatic extension of $\mathbf{R}$ is structurally complete, except for classical propositional logic [4]. The situation for $\mathbf{R^t}$ is very different and is the subject of ongoing algebraic investigation by the present authors.

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