Quasi injectivity of partially ordered acts

M.Mehdi Ebrahimi and Mahdieh Yavari*

[1] Department of Mathematics, Shahid Beheshti University, G.C., Tehran 19839, Iran.
m-ebrahimi@sbu.ac.ir
[2] Department of Mathematics, Shahid Beheshti University, G.C., Tehran 19839, Iran.
m_yavari@sbu.ac.ir

1 Introduction and Preliminaries

It is well-known that injective objects play a fundamental role in many branches of mathematics. The question whether a given category has injective objects has been investigated for many categories. As for posets, Banaschewski [1] proves that complete posets are exactly $\mathcal{E}$-injective posets (injective with respect to order-embeddings), and Sikorski [4] shows the same result for injective Boolean algebras (see also [2, 3]).

In this paper, we study quasi injectivity in the category of (right) actions of a partially ordered monoid on partially ordered sets (**Pos**-S) with respect to embeddings ($\mathcal{E}$-quasi injectivity). First, we study the relation between $\mathcal{E}$-injectivity, $\mathcal{E}$-quasi injectivity, and completeness in **Pos**-S. Then, we show when a θ-extension of an $\mathcal{E}$-quasi injective S-poset A ($A \oplus \{\theta\}$, which is obtained by adjoining a zero top element θ to A) is K_θ-quasi injective. Note that an S-poset $A \oplus \{\theta\}$ is called K_θ-quasi injective if for every sub S-poset B of $A \oplus \{\theta\}$, any S-poset map $f : B \to A \oplus \{\theta\}$, with $f^{-1}(\theta) = \overleftarrow{f}(\theta) = \{b \in B : f(b) = \theta\} \neq \emptyset$, can be extended to $\bar{f} : A \oplus \{\theta\} \to A \oplus \{\theta\}$. Finally, we study the relation between $\mathcal{E}$-injectivity, $\mathcal{E}$-quasi injectivity, and completeness in some useful subcategories of **Pos**-S.

Definition 1.1. Let $\mathcal{M}$ be a class of monomorphisms in a category $\mathcal{C}$. An object $A \in \mathcal{C}$ is called

1. $\mathcal{M}$-*injective* if for each $\mathcal{M}$-*morphism* $m : B \to C$ and any morphism $f : B \to A$ there exists a morphism $\bar{f} : C \to A$ such that $\bar{f}m = f$,
2. $\mathcal{M}$-*quasi injective* if for each $\mathcal{M}$-morphism $m : B \to A$ and any morphism $f : B \to A$ there exists a morphism $\bar{f} : A \to A$ which extends f,
3. $\mathcal{M}$-*absolute retract* if it is a retract of each of its $\mathcal{M}$-extensions; that is, for each $\mathcal{M}$-morphism $m : A \to C$ there exists a morphism $\bar{f} : C \to A$ such that $\bar{f}m = id_A$, in which case $\bar{f}$ is said to be a *retraction*.

2 Main Results

Remark 2.1. It is clear that every $\mathcal{E}$-injective S-poset is $\mathcal{E}$-quasi injective. But the converse is not necessarily true. (In Theorem 2.4 (below) we give conditions under which the converse is also true.)

Remark 2.2. 1. Every complete poset with identity action is $\mathcal{E}$-quasi injective in the category **Pos**-S.

2. The action on an $\mathcal{E}$-quasi injective S-poset need not be identity.
3. An $\mathcal{E}$-quasi injective S-poset is not necessarily complete as a poset.
4. There exists complete S-poset which is not $\mathcal{E}$-quasi injective.

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Proposition 2.3. *There exists no pomonoid S over which all S-posets are $\mathcal{E}$-quasi injective.*

Theorem 2.4. *An $\mathcal{E}$-quasi injective S-poset A is $\mathcal{E}$-injective if and only if A has a zero element and $A \times \bar{A}^{(S)}$ is $\mathcal{E}$-quasi injective ($\bar{A}$ is the Dedekind-MacNeille completion of A).*

Definition 2.5. Let A be an S-act. A subset B of A is called *consistent* if for each $a \in A$ and $s \in S$, $as \in B$ implies $a \in B$. We call a consistent subact an *S-filter*.

Theorem 2.6. *Let A be an $\mathcal{E}$-quasi injective S-poset. Also, assume that $f : B \to A \oplus \{\theta\}$ is an S-poset map, where B is a sub S-poset of $A \oplus \{\theta\}$ and $\overleftarrow{f}(\theta) \neq \emptyset$. Then there exists an S-filter $\tilde{A}$ of $A \oplus \{\theta\}$ which is upward closed in $A \oplus \{\theta\}$, $\overleftarrow{f}(\theta) \subseteq \tilde{A}$, and $\tilde{A} \cap \{b \in B : f(b) \neq \theta\} = \emptyset$ if and only if there exists an S-poset map $\bar{f} : A \oplus \{\theta\} \to A \oplus \{\theta\}$ which extends f.*

Corollary 2.7. *Let A be an $\mathcal{E}$-quasi injective S-poset. If for each S-poset map $f : B \to A \oplus \{\theta\}$, where B is a sub S-poset of $A \oplus \{\theta\}$ and $\overleftarrow{f}(\theta) \neq \emptyset$, we have*

$$\tilde{A} = \{a \in A \oplus \{\theta\} : \exists s \in S, as \in \uparrow \overleftarrow{f}(\theta)\}$$

is an S-filter of $A \oplus \{\theta\}$, then $A \oplus \{\theta\}$ is a K_θ-quasi injective S-poset.

Corollary 2.8. *Suppose S is a pomonoid with any one of the following properties*:

(1) *$\forall s \in S$, $\exists t \in S$, $st \leq e$ (e is the identity element of S).*

(2) *$\forall s \in S$, $s^2 \leq e$.*

(3) *S is a pogroup.*

(4) *$\top_S = e$ (S has the top element $\top_S$).*

(5) *S is a right zero semigroup with an adjoined identity.*

If A is an $\mathcal{E}$-quasi injective S-poset then $A \oplus \{\theta\}$ is K_θ-quasi injective in **Pos**-S.

Definition 2.9. An S-poset A is called *strong reversible* if for every $s \in S$ there exists $t \in S$ such that $ast = ats = a$ for all $a \in A$. Also, an S-poset A is *square reversible* if for every $a \in A$ and $s \in S$, we have $as^2 = a$. So, we have the category **SR-Pos**-S (**SQ-Pos**-S) of all strong (square) reversible S-posets and S-poset maps between them.

Theorem 2.10. *A strong (square) reversible S-poset is $\mathcal{E}$-quasi injective in* **SR-Pos**-S (**SQ-Pos**-S) if it is complete.

Theorem 2.11. *Let A be a strong (square) reversible S-poset. Then the following are equivalent*:

(i) *A is $\mathcal{E}$-injective in* **SR-Pos**-S (**SQ-Pos**-S).

(ii) A is $\mathcal{E}$-absolute retract in **SR-Pos**-S (**SQ-Pos**-S).

(iii) A is complete.

References

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