Modal logics over finite residuated lattices

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Several publications in the literature address the study of modal expansions of many-valued logics (see eg. [3, 2] [1], [4], [5]), and it is the aim of the current work to contribute to the better understanding of this topic. In particular, the framework developed in [1], which focused on modal logics defined from classes of Kripke models evaluated over finite residuated lattices, proposed several interesting open problems, and along the following lines we solve some of them.

In [1] we can find an standard definition of the Kripke models of a finite integral bounded commutative residuated lattice $\mathbf{A}$ (simply called finite residuated lattices in the rest of the abstract). There, the authors provide an axiomatic system for the local consequence relation arising from the models of $\mathbf{A}^c$ (the algebra $\mathbf{A}$ expanded with canonical constants[1]), while it is left as open problem the formulation of the corresponding system for the analogous global consequence relation.[2] On the other hand, the modal logics studied in [1] consider only the $\Box$ modal operator, interpreted in a world of the model, as usual, by $e(v, \Box\varphi) = \bigwedge_{w\in W} Rvw \to e(w,\varphi)$. It is well known that, over residuated lattices the dual $\Diamond$ operator, for $e(v, \Diamond\varphi) = \bigwedge_{w\in W} Rvw \odot e(w,\varphi)$, is in general no longer definable from $\Box$, since the negation needs not to be involutive.

We wish to address the previous topics towards the full characterization, and subsequent understanding, applicability and possible generalisation of modal many-valued logics.

For $\mathbf{B}$ a finite residuated lattice (with or without canonical constants) consider the following logics, all of them defined from classes of Kripke models as usual:

- $\Vdash^l_{\Box\mathbf{B}}$ and $\Vdash^l_{\Diamond\Box\mathbf{B}}$: the local consequence relation over the Kripke models of $\mathbf{B}$, with only $\Box$ and with both $\Box$ and $\Diamond$ operators,
- $\Vdash^g_{\Box\mathbf{B}}$ and $\Vdash^g_{\Diamond\Box\mathbf{B}}$: the global consequence relation over the Kripke models of $\mathbf{B}$, with only $\Box$ and with both $\Box$ and $\Diamond$ operators.

Let $\vdash^l_{\Box\mathbf{A}^c}$ denote the axiomatic system for $\Vdash^l_{\Box\mathbf{A}^c}$ provided in [1, Th. 4.11] (called there $\mathbf{\Lambda}(l, \mathbf{Fr}, \mathbf{A}^c)$). As far as we know, all the other logics from the previous list have not been axiomatized in the literature, for arbitrary $\mathbf{B}$. We can first exhibit recursively enumerable axiomatic systems for $\Vdash^l_{\Diamond\Box\mathbf{A}^c}$ and for both global logics arising from algebras with constants.

1. *Let $\vdash^l_{\Diamond\Box\mathbf{A}^c}$ be the axiomatic system $\vdash^l_{\Box\mathbf{A}^c}$ extended with $\Box(\varphi \to \overline{c}) \leftrightarrow (\Diamond\varphi \to \overline{c})$ for each $c \in A$. Then for any set of formulas Γ, φ it holds that*
$$\Gamma \vdash^l_{\Diamond\Box\mathbf{A}^c} \varphi \iff \Gamma \Vdash^l_{\Diamond\Box\mathbf{A}^c} \varphi$$
2. *Let $\vdash^g_{\Box\mathbf{A}^c}$ be the axiomatic system $\vdash^l_{\Box\mathbf{A}^c}$ expanded with the rule $(\mathtt{Mon}) : \varphi \to \psi \rhd \Box\varphi \to \Box\psi$. Similarly, let $\vdash^g_{\Diamond\Box\mathbf{A}^c}$ be the axiomatic system $\vdash^l_{\Diamond\Box\mathbf{A}^c}$ expanded with the rule $(\mathtt{Mon})$. Then for any set of formulas Γ, φ*
$$\Gamma \vdash^g_{\Box\mathbf{A}^c} \varphi \iff \Gamma \Vdash^g_{\Box\mathbf{A}^c} \varphi \quad \textit{and} \quad \Gamma \vdash^g_{\Diamond\Box\mathbf{A}^c} \varphi \iff \Gamma \Vdash^g_{\Diamond\Box\mathbf{A}^c} \varphi$$

[1] That is, with one additional constant symbol for each one of its elements, interpreted in the natural way.

[2] We are interested here in the models where the accessibility relation is evaluated in A. In [1] it is presented an axiomatization of the global consequence of the smaller class of models with $\{0, 1\}$-valued accessibility relation. It is worth to remark that the usual `K` axiom holds in this restricted class of models, while it does not in the general framework.

In order to prove the previous completeness results, finitarity of the propositional logic (due to finiteness of the algebra) and presence of constants are crucial. Nevertheless, resorting to those properties it is not hard to prove the Truth Lemma for the usual Canonical Models (both for the local and global logics). We remark here for the interested reader a cornerstone property from which the Truth Lemma follows easily, provable for all the previous canonical models:[3]

For any formula ψ and any $h \in W^c$, if $R^c hg \leq g(\psi)$ for all $g \in W^c$ then $h(\Box\psi) = 1$.

The relation between the local and global modal logics of the class of frames over an algebra $\mathbf{A}$ (and over its corresponding $\mathbf{A}^c$) is now easy to understand. First, from the previous completeness result is immediate that the logic $\Vdash^g_{\Box\mathbf{A}^c}$ ($\Vdash^g_{\Diamond\Box\mathbf{A}^c}$) is the smallest consequence relation containing $\Vdash^l_{\Box\mathbf{A}^c}$ (respectively, $\Vdash^l_{\Diamond\Box\mathbf{A}^c}$) and closed under ($\mathtt{Mon}$). On the other hand, by the definition of the modal logics arising from a class of models, is clear that for any set of formulas Γ, φ *without (non-trivial) constant symbols* it holds that

$$\Gamma \Vdash^*_{\mathtt{M}\mathbf{A}} \varphi \quad \Longleftrightarrow \quad \Gamma \Vdash^*_{\mathtt{M}\mathbf{A}^c} \varphi$$

for $* \in \{l, g\}$ and $\mathtt{M} \in \{\Box, \Diamond\Box\}$.

Even if we do not have a syntactical characterization of the modal logics arising from Kripke models over residuated lattices without canonical constants, the previous relation provides interesting information about these logics. A first immediate observation is that the logics $\Vdash^*_{\mathtt{M}\mathbf{A}}$ are finitary, since so are their corresponding versions over $\mathbf{A}^c$. Finitarity of both, the logic and the rule ($\mathtt{Mon}$), provide (*via* eg. Zorn's Lemma) a very simple characterization of $[\Vdash^l_{\mathtt{M}\mathbf{A}}]^{(\mathtt{Mon})}$, the minimum consequence relation expanding $\Vdash^l_{\mathtt{M}\mathbf{A}}$ closed under ($\mathtt{Mon}$). From there, it is easy to prove that for any Γ, φ without non-trivial constant symbols

$$\Gamma[\Vdash^l_{\mathtt{M}\mathbf{A}}]^{(\mathtt{Mon})}\varphi \quad \Longleftrightarrow \quad \Gamma[\Vdash^l_{\mathtt{M}\mathbf{A}^c}]^{(\mathtt{Mon})}\varphi$$

After the previous observations and using the completeness results stated above for logics over lattices with canonical constants, we can prove the following chain of equivalences:

$$\Gamma \Vdash^g_{\mathtt{M}\mathbf{A}} \varphi \quad \Longleftrightarrow \quad \Gamma \Vdash^g_{\mathtt{M}\mathbf{A}^c} \varphi \quad \Longleftrightarrow \quad \Gamma[\Vdash^l_{\mathtt{M}\mathbf{A}^c}]^{(\mathtt{Mon})}\varphi \quad \Longleftrightarrow \quad \Gamma[\Vdash^l_{\mathtt{M}\mathbf{A}}]^{(\mathtt{Mon})}\varphi.$$

This answers positively the open question (4) from [1], on whether the global deduction is the minimum closure operator containing the local deduction one and closed under the ($\mathtt{Mon}$), both for finite residuated lattices with and without canonical constants.

References

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[3] This property is applicable also in some cases based on infinite residuated lattices, see eg. [6].