Topology from enrichment: the curious case of partial metrics

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Following Fréchet [1], a metric space (X, d) is a set X together with a real-valued function d on $X \times X$ such that the following axioms hold:

[M0] $d(x, y) \geq 0$,
[M1] $d(x, y) + d(y, z) \geq d(x, z)$,
[M2] $d(x, x) = 0$,
[M3] if $d(x, y) = 0 = d(y, x)$ then $x = y$,
[M4] $d(x, y) = d(y, x)$,
[M5] $d(x, y) \neq +\infty$.

The categorical content of this definition, as first observed by Lawvere [6], is that the extended real interval $[0, \infty]$ underlies a commutative quantale $([0, \infty], \bigwedge, +, 0)$, so that a "generalised metric space" (i.e. a structure as above, minus the axioms M3-M4-M5) is exactly a category enriched in that quantale. It was furthermore shown in [4] that to any category enriched in a commutative quantale one can associate a closure operator on its collection of objects. For a metric space (X, d), viewed as an $[0, \infty]$-enriched category, that "categorical closure" on X coincides precisely with the metric (topological) closure defined by d. And Lawvere [6] famously reformulated the Cauchy completeness of a metric space in terms of adjoint distributors.

More recently, see e.g. [7], the notion of a partial metric space (X, p) has been proposed to mean a set X together with a real-valued function p on $X \times X$ satisfying the following axioms:

[P0] $p(x, y) \geq 0$,
[P1] $p(x, y) + p(y, z) - p(y, y) \geq p(x, z)$,
[P2] $p(x, y) \geq p(x, x)$,
[P3] if $p(x, y) = p(x, x) = p(y, y) = p(y, x)$ then $x = y$,
[P4] $p(x, y) = p(y, x)$,
[P5] $p(x, y) \neq +\infty$.

The categorical content of *this* definition was discovered in two steps: first, Höhle and Kubiak [5] showed that there is a particular quantaloid of positive real numbers, such that categories enriched in that quantaloid correspond to ("generalised") partial metric spaces; and second, we realised in [8] that Höhle and Kubiak's quantaloid of real numbers is actually a universal construction on Lawvere's quantale of real numbers: namely, the quantaloid $\mathcal{D}[0, \infty]$ of diagonals in $[0, \infty]$.

In this talk we shall show how every small quantaloid-enriched category has a canonical closure operator on its set of objects: this makes for a functor from quantaloid-enriched categories to closure spaces. Under mild necessary-and-sufficient conditions on the base quantaloid, this functor lands in the category of topological spaces; and an involutive quantaloid is Cauchy-bilateral (a property discovered earlier in the context of distributive laws [2]) if and only if the closure on any enriched category is identical to the closure on its symmetrisation. As this now applies to metric spaces and partial metric spaces alike, we demonstrate how these general categorical constructions produce the "correct" definitions of convergence and Cauchyness of sequences in generalised partial metric spaces. Finally we describe the Cauchy-completion (and,

if time premits, also the Hausdorff contruction and exponentiability) of a partial metric space, again by application of general quantaloid-enriched category theory.

This talk is based on a joint paper with Dirk Hofmann [3].

References

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