A Vietoris functor for bispaces and d-frames

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Vietoris topology [12], hyperspace, powerlocale or powerdomain are many names for the same phenomenon. The scope of its applications ranges from semantics of programming languages [1, 10], coalgebraic logic [7, 8] to modal logic [2, 7, 11, 9]. Also, in Abramsky's *"Domain theory in logical form"* [1], a Vietoris construction has been an important tool for establishing a connection between syntax and semantics. An example of another such situation is the Jónsson-Tarski duality [5]. In modern parlance, we have an endofunctor $\mathbb{V}$ on Stone spaces and an endofunctor on Boolean algebras $\mathbb{M}$:

$$\begin{tikzcd} \mathbf{Stone} \arrow[loop left, "\mathbb{V}"] \arrow[rr, bend left, "\mathsf{Clp}"] & & \mathbf{Bool} \arrow[loop right, "\mathbb{M}"] \arrow[ll, bend left, "\mathsf{spec}"] \end{tikzcd}$$

Moreover, the duality of Stone spaces and Boolean algebras extends to a duality of $\mathbb{V}$-coalgebras and $\mathbb{M}$-algebras. The category of $\mathbb{V}$-coalgebras is isomorphic to the category of descriptive general Kripke frames and the category of $\mathbb{M}$-algebras is isomorphic to the category of modal Boolean algebras.

Jónnson-Tarski duality is an instance of a more general picture where we can substitute **Stone** by a suitable category of spaces (modelling semantics) and **Bool** by a suitable category of algebras (modelling syntax) such that those categories are dually equivalent and some interconnected powerconstructions $\mathbb{V}$ and $\mathbb{M}$ still exist. Other examples, where we can replace the base categories, include Priestley spaces and distributive lattices, or compact Hausdorff spaces and compact regular frames.

It was a beautiful insight by Jung and Moshier that all the dualities mentioned in the previous paragraph, and many more, sit in the duality between compact regular bitopological spaces **biKReg** and compact regular d-frames **d-KReg** [6]. Here d-frames are algebraic duals of bitopological spaces in the same way as frames[1] are algebraic duals of (ordinary) spaces. We give Vietoris endofunctors $\mathbb{W}$ and $\mathbb{M}^d$ which are generalisations of the corresponding Vietoris constructions for Stone, Priestley and frame dualities mentioned above.

$$\begin{tikzcd} \mathbf{biKReg} \arrow[loop left, "\mathbb{W}"] \arrow[rr, bend left, "\Omega"] & & \mathbf{d\text{-}KReg} \arrow[loop right, "\mathbb{M}^d"] \arrow[ll, bend left, "\Sigma"] \end{tikzcd}$$

The construction

On the semantic side, we have the endofunctor $\mathbb{W}\colon \mathbf{biKReg} \to \mathbf{biKReg}$. Similarly to the Vietoris endofunctor for spaces or domains, the points of $\mathbb{W}(X;\tau_+,\tau_-)$ are compact *convex* subsets of X

[1] *Frames* are complete lattices satisfying the equation: $a \wedge (\bigvee_i b_i) = \bigvee_i (a \wedge b_i)$.

and subbases of the topologies of $\mathbb{W}(X)$ are sets $\{\boxtimes U_+, \oplus U_+ : U_+ \in \tau_+\}$ and $\{\boxtimes U_-, \oplus U_- : U_- \in \tau_-\}$ where

$$K \in \boxtimes U \text{ iff } K \subseteq U \quad \text{and} \quad K \in \oplus U \text{ iff } K \cap U \neq \emptyset.$$

On the algebraic side we have d-frames, i.e. structures of the form $(L_+, L_-; \mathsf{con}, \mathsf{tot})$ where L_+ and L_- are frames corresponding to the two topologies and $\mathsf{con} \subseteq L_+ \times L_-$ is a relation which captures when two abstract opens are disjoint from each other and, similarly, $\mathsf{tot} \subseteq L_+ \times L_-$ representing when two abstract opens cover the whole space. The endofunctor $\mathbb{M}^d\colon \mathbf{d\text{-}Frm} \to \mathbf{d\text{-}Frm}$ is computed as follows

$$\mathbb{M}^d\colon\ (L_+, L_-; \mathsf{con}, \mathsf{tot}) \quad\longmapsto\quad (\mathbb{M}^{\mathbf{Frm}} L_+,\ \mathbb{M}^{\mathbf{Frm}} L_-;\ \mathsf{con}^{\mathbb{M}}, \mathsf{tot}^{\mathbb{M}}).$$

Here, $\mathbb{M}^{\mathbf{Frm}}$ is the Johnstone's powerlocale construction for frames [4]. To describe the consistency and totality relations we need to develop a free construction of a d-frame and then $\mathsf{con}^{\mathbb{M}}$ and $\mathsf{tot}^{\mathbb{M}}$ can be given by a set of generators. Similarly as in frames, $\mathbb{M}^d$ is comonadic and we also have the following familiar result:

Theorem. *Let $\mathcal{L}$ be a d-frame. If $\mathcal{L}$ is regular, zero-dimensional or compact regular then also $\mathbb{M}^d\mathcal{L}$ is.*

Thanks to this, we can restrict $\mathbb{M}^d$ to an endofunctor $\mathbb{M}^d\colon \mathbf{d\text{-}KReg} \to \mathbf{d\text{-}KReg}$ and formalise the connection between $\mathbb{W}$ and $\mathbb{M}^d$. The first step is to investigate the spectrum bispace $\Sigma\mathbb{M}^d(\mathcal{L})$. It turns out that its points have a very natural description. Namely, they are in bijection with the set of $\alpha \in L_+ \times L_-$ such that

(A+) $\quad \forall u_+ \in L_+\colon$ if $(\alpha_+ \vee u_+, \alpha_-) \in \mathsf{tot}$ then $(u_+, \alpha_-) \in \mathsf{tot}$

(A−) $\quad \forall u_- \in L_-\colon$ if $(\alpha_+, \alpha_- \vee u_-) \in \mathsf{tot}$ then $(\alpha_+, u_-) \in \mathsf{tot}$

We can now prove the main result:

Theorem. *Let $\mathcal{L}$ be a compact regular d-frame. Then, $\mathbb{W}\Sigma(\mathcal{L}) \cong \Sigma\mathbb{M}^d(\mathcal{L})$. Moreover, this bi-homeomorphism is natural in $\mathcal{L}$.*

Thanks to this, we can lift the dual equivalence of categories **biKReg** and **d-KReg** to a dual equivalence of the category of $\mathbb{W}$-coalgebras and the category of $\mathbb{M}^d$-algebras.

As mentioned above, $\mathbb{W}$ is a generalisation of the corresponding Vietoris constructions for Stone spaces, Priestley spaces or compact regular spaces. Similarly, $\mathbb{M}^d$ is a generalisation of the constructions for Boolean algebras, distributive lattices and compact regular frames.

Another duality that embeds into the duality $\mathbf{biKReg}^{\mathrm{op}} \cong \mathbf{d\text{-}KReg}$ is the duality between the category of stably compact spaces (i.e. compact Hausdorff ordered spaces) and the category of strong proximity lattices. In fact, those categories are equivalent to **biKReg** and **d-KReg**, respectively. Although a Vietoris construction is known for stably compact spaces [3], $\mathbb{M}^d$ is (to our knowledge) the first algebraic counterpart for it. Moreover, the free construction we developed for d-frame Vietoris functor is the first free construction of a d-frame.

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