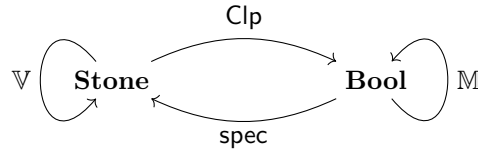


A Vietoris functor for bispaces and d-frames

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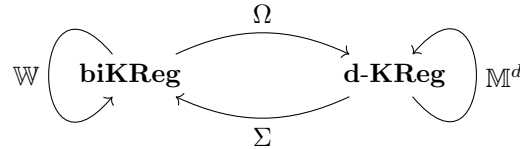
Vietoris topology [12], hyperspace, powerlocale or powerdomain are many names for the same phenomenon. The scope of its applications ranges from semantics of programming languages [1, 10], coalgebraic logic [7, 8] to modal logic [2, 7, 11, 9]. Also, in Abramsky’s “*Domain theory in logical form*” [1], a Vietoris construction has been an important tool for establishing a connection between syntax and semantics. An example of another such situation is the Jónsson-Tarski duality [5]. In modern parlance, we have an endofunctor \mathbb{V} on Stone spaces and an endofunctor on Boolean algebras \mathbb{M} :



Moreover, the duality of Stone spaces and Boolean algebras extends to a duality of \mathbb{V} -coalgebras and \mathbb{M} -algebras. The category of \mathbb{V} -coalgebras is isomorphic to the category of descriptive general Kripke frames and the category of \mathbb{M} -algebras is isomorphic to the category of modal Boolean algebras.

Jónsson-Tarski duality is an instance of a more general picture where we can substitute **Stone** by a suitable category of spaces (modelling semantics) and **Bool** by a suitable category of algebras (modelling syntax) such that those categories are dually equivalent and some interconnected power-constructions \mathbb{V} and \mathbb{M} still exist. Other examples, where we can replace the base categories, include Priestley spaces and distributive lattices, or compact Hausdorff spaces and compact regular frames.

It was a beautiful insight by Jung and Moshier that all the dualities mentioned in the previous paragraph, and many more, sit in the duality between compact regular bitopological spaces **biKReg** and compact regular d-frames **d-KReg** [6]. Here d-frames are algebraic duals of bitopological spaces in the same way as frames¹ are algebraic duals of (ordinary) spaces. We give Vietoris endofunctors \mathbb{W} and \mathbb{M}^d which are generalisations of the corresponding Vietoris constructions for Stone, Priestley and frame dualities mentioned above.



The construction

On the semantic side, we have the endofunctor $\mathbb{W}: \mathbf{biKReg} \rightarrow \mathbf{biKReg}$. Similarly to the Vietoris endofunctor for spaces or domains, the points of $\mathbb{W}(X; \tau_+, \tau_-)$ are compact *convex* subsets of X

¹Frames are complete lattices satisfying the equation: $a \wedge (\bigvee_i b_i) = \bigvee_i (a \wedge b_i)$.

and subbases of the topologies of $\mathbb{W}(X)$ are sets $\{\boxtimes U_+, \boxplus U_+ : U_+ \in \tau_+\}$ and $\{\boxtimes U_-, \boxplus U_- : U_- \in \tau_-\}$ where

$$K \in \boxtimes U \text{ iff } K \subseteq U \quad \text{and} \quad K \in \boxplus U \text{ iff } K \cap U \neq \emptyset.$$

On the algebraic side we have d-frames, i.e. structures of the form $(L_+, L_-; \text{con}, \text{tot})$ where L_+ and L_- are frames corresponding to the two topologies and $\text{con} \subseteq L_+ \times L_-$ is a relation which captures when two abstract opens are disjoint from each other and, similarly, $\text{tot} \subseteq L_+ \times L_-$ representing when two abstract opens cover the whole space. The endofunctor $\mathbb{M}^d: \mathbf{d-Frm} \rightarrow \mathbf{d-Frm}$ is computed as follows

$$\mathbb{M}^d: (L_+, L_-; \text{con}, \text{tot}) \longmapsto (\mathbb{M}^{\mathbf{Frm}} L_+, \mathbb{M}^{\mathbf{Frm}} L_-; \text{con}^{\mathbb{M}}, \text{tot}^{\mathbb{M}}).$$

Here, $\mathbb{M}^{\mathbf{Frm}}$ is the Johnstone's powerlocale construction for frames [4]. To describe the consistency and totality relations we need to develop a free construction of a d-frame and then $\text{con}^{\mathbb{M}}$ and $\text{tot}^{\mathbb{M}}$ can be given by a set of generators. Similarly as in frames, \mathbb{M}^d is comonadic and we also have the following familiar result:

Theorem. *Let \mathcal{L} be a d-frame. If \mathcal{L} is regular, zero-dimensional or compact regular then also $\mathbb{M}^d \mathcal{L}$ is.*

Thanks to this, we can restrict \mathbb{M}^d to an endofunctor $\mathbb{M}^d: \mathbf{d-KReg} \rightarrow \mathbf{d-KReg}$ and formalise the connection between \mathbb{W} and \mathbb{M}^d . The first step is to investigate the spectrum bispace $\Sigma \mathbb{M}^d(\mathcal{L})$. It turns out that its points have a very natural description. Namely, they are in bijection with the set of $\alpha \in L_+ \times L_-$ such that

$$\begin{aligned} (\text{A}+) \quad & \forall u_+ \in L_+: \text{ if } (\alpha_+ \vee u_+, \alpha_-) \in \text{tot} \text{ then } (u_+, \alpha_-) \in \text{tot} \\ (\text{A}-) \quad & \forall u_- \in L_-: \text{ if } (\alpha_+, \alpha_- \vee u_-) \in \text{tot} \text{ then } (\alpha_+, u_-) \in \text{tot} \end{aligned}$$

We can now prove the main result:

Theorem. *Let \mathcal{L} be a compact regular d-frame. Then, $\mathbb{W}\Sigma(\mathcal{L}) \cong \Sigma \mathbb{M}^d(\mathcal{L})$. Moreover, this bi-homeomorphism is natural in \mathcal{L} .*

Thanks to this, we can lift the dual equivalence of categories \mathbf{biKReg} and $\mathbf{d-KReg}$ to a dual equivalence of the category of \mathbb{W} -coalgebras and the category of \mathbb{M}^d -algebras.

As mentioned above, \mathbb{W} is a generalisation of the corresponding Vietoris constructions for Stone spaces, Priestley spaces or compact regular spaces. Similarly, \mathbb{M}^d is a generalisation of the constructions for Boolean algebras, distributive lattices and compact regular frames.

Another duality that embeds into the duality $\mathbf{biKReg}^{\text{op}} \cong \mathbf{d-KReg}$ is the duality between the category of stably compact spaces (i.e. compact Hausdorff ordered spaces) and the category of strong proximity lattices. In fact, those categories are equivalent to \mathbf{biKReg} and $\mathbf{d-KReg}$, respectively. Although a Vietoris construction is known for stably compact spaces [3], \mathbb{M}^d is (to our knowledge) the first algebraic counterpart for it. Moreover, the free construction we developed for d-frame Vietoris functor is the first free construction of a d-frame.

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