

# On exponentiability in quantale-enriched categories

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## 1 Introduction

The purpose of this work is to present results on the existence of “function spaces” in categories of quantale-enriched categories, with particular emphasis on *generalized metric spaces* and *generalized probabilistic metric spaces* and their corresponding *non-expansive maps* (see [7] and [6]), that is, *V-categories* and *V-functors* when *V* is respectively the quantale of the *complete half-real line equipped with addition* and the *quantale of distribution functions*, but focusing also on categories enriched in the unit interval equipped with a *continuous t-norm*.

Most of the material presented is part of joint work with Dirk Hofmann that is published in [2] and [3] (see also [4]).

## 2 Exponentiable quantale-enriched categories

Given a quantale  $(V, \otimes, k)$ , it is well-known that the category  $V\text{-Cat}$  of enriched *V-categories* and *V-functors* is *closed*, that is, for each *V-category* *X* the functor  $- \otimes X : V\text{-Cat} \rightarrow V\text{-Cat}$  induced in *V-Cat* by the tensor  $\otimes$  in *V* has a right adjoint (see [7]). Here we concentrate on *cartesian closedness* of *V-Cat*, that is, on the existence of a right adjoint  $(-)^X : V\text{-Cat} \rightarrow V\text{-Cat}$  to the functor  $- \times X : V\text{-Cat} \rightarrow V\text{-Cat}$  for each *V-category* *X*. This is not always the case, and in fact this ends up on the existence of a convenient *V-category* structure on the set  $Y^X$  of *V-functors* from *X* to *Y* (= exponential of *Y* with exponent *X*), for every *V-category* *Y*.

We will also pose this problem more generally, studying instead the existence of exponentials in the comma categories  $(V\text{-Cat}) \downarrow Y$ . Inspired by known results on exponentiability for continuous maps between topological spaces (see [5] and [1]), we prove in particular that *every proper and every étale V-functor is exponentiable in V-Cat*.

## 3 Final remarks

This study leads also to some interesting open problems on the properties of the quantales involved, namely on the quantale of distribution functions.

## References

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