First-order interpolation may be derived from propositional interpolation

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Following the ground breaking results of Maksimova [6] many families of propositional logics have been classified w.r.t. the interpolation property. However, on first-order level, the knowledge about interpolation is restricted. Moreover, it is not known which of the seven interpolating intermediary propositional logics [5] admit first-order interpolation (first-order infinitely-valued Gödel logic $G_{[0,1]}$ is the most notable example).

This lecture develops a general methodology to connect propositional and first-order interpolation. The construction of the first-order interpolant follows this procedure:

$$\left.\begin{array}{r}\text{existence of suitable Skolemizations } + \\ \text{existence of Herbrand expansions } + \\ \text{propositional interpolant}\end{array}\right\} \rightarrow \begin{array}{c}\text{first-order} \\ \text{interpolation.}\end{array}$$

This methodology is realized for lattice-based finitely-valued logics, the top element representing true and can be extended to (fragments of) infinitely-valued logics.

The construction of the first-order interpolant from the propositional interpolant follows this procedure:

1. Develop a validity equivalent Skolemization replacing all strong quantifiers (negative existential or positive universal quantifiers) in the valid formula $A \supset B$ to obtain the valid formula $A_1 \supset B_1$.

2. Construct a valid Herbrand expansion $A_2 \supset B_2$ for $A_1 \supset B_1$. Occurrences of $\exists x B(x)$ and $\forall x A(x)$ are replaced by suitable finite disjunctions $\bigvee B(t_i)$ and conjunctions $\bigwedge B(t_i)$, respectively.

3. Interpolate the propositionally valid formula $A_2 \supset B_2$ with the propositional interpolant I^*: $A_2 \supset I^*$ and $I^* \supset B_2$ are propositionally valid.

4. Reintroduce weak quantifiers to obtain valid formulas $A_1 \supset I^*$ and $I^* \supset B_1$.

5. Eliminate all function symbols and constants not in the common language of A_1 and B_1 by introducing suitable quantifiers in I^* (note that no Skolem functions are in the common language, therefore they are eliminated). Let I be the result.

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*The first author discussed the problem of deciding the admissibility of interpolation in first-order logics on the basis of the admissibility interpolation in propositional logics with Petr Hájek who suggested that proof-theoretic approaches might help to overcome the lack of algebraization of first-order logics.

6. I is an interpolant for $A_1 \supset B_1$. $A_1 \supset I$ and $I \supset B_1$ are Skolemizations of $A \supset I$ and $I \supset B$. Therefore I is an interpolant of $A \supset B$.

This methodology is realized for lattice-based finitely-valued logics and can be extended to (fragments of) infinitely-valued logics (more precisely to fragments of first-order infinitely-valued Gödel logic).

Consider Gödel logic $G_{[0,1]}$, the logic of all linearly ordered Kripke frames with constant domains. Its connectives can be interpreted as functions over the real interval $[0, 1]$ as follows: $\perp$ is the logical constant for 0, $\vee, \wedge, \exists, \forall$ are defined as *maximum*, *minimum*, *supremum*, *infimum*, respectively. $\neg A$ is an abbreviation for $A \to \perp$ and $\to$ is defined as

$$u \to v = \begin{cases} 1 & u \leq v \\ v & \text{else} \end{cases}$$

The weak quantifier fragment of $G_{[0,1]}$ admits Herbrand expansions. This follows from cut-free proofs in hypersequent calculi [1, 2, 3]. This can be easily shown by proof transformation steps in the hypersequent calculus. Indeed, we can transform proofs by eliminating weak quantifier inferences:

i If there is an occurrence of an $\exists$ introduction, we select all formulas A_i that correspond to this inference and eliminate the $\exists$ introduction by the use of $\bigvee_i A_i$.

ii If there is an occurrence of a $\forall$ introduction, we select all formulas B_i that correspond to this inference and eliminate the $\forall$ introduction by the use of $\bigwedge_i B_i$.

With this procedure we do not infer weak quantifiers and combine the disjunctions/conjunctions to accommodate contractions. Propositional Gödel logic interpolates and therefore the weak quantifier fragment of $G_{[0,1]}$ interpolates, too.

The fragment $A \supset B$, A, B prenex also interpolates: Skolemize as in classical logic, construct a Herbrand expansion, interpolate, go back to the Skolem form and use an immediate analogy of the 2nd ε-theorem [4] to go back to the original formulas.

References

[1] Baaz, M., Ciabattoni, A., and Fermüller, C. G. Hypersequent calculi for Gödel logics – a survey. *Journal of Logic and Computation 13*, 6 (2003), 835–861.

[2] Ciabattoni, A. Hypersequent calculi for some intermediate logics with bounded Kripke models. *Journal of Logic and Computation 11*, 2 (2001), 283–294.

[3] Ciabattoni, A., Ramanayake, R., and Wansing, H. Hypersequent and display calculi–a unified perspective. *Studia Logica 102*, 6 (2014), 1245–1294.

[4] Hilbert, D., and Bernays, P. Grundlagen der Mathematik.

[5] Maksimova, L. L. Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-boolean algebras. *Algebra and Logic 16*, 6 (1977), 427–455.

[6] Maksimova, L. L. Interpolation properties of superintuitionistic logics. *Studia Logica 38*, 4 (1979), 419–428.