# Antistructural completeness in propositional logics

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In this contribution, we shall investigate the notion of an antistructural completion $\alpha\mathcal{L}$ of a propositional logic $\mathcal{L}$, which is in a natural sense dual to the well-known notion of a structural completion of a logic, and provide several equivalent characterizations of such completions under some mild conditions on the logic in question.

Recall that the *structural completion* of a logic $\mathcal{L}$ is the largest logic $\sigma\mathcal{L}$ which has the same theorems as $\mathcal{L}$ (see [2]). A logic $\mathcal{L}$ is then called *structurally complete* if $\sigma\mathcal{L} = \mathcal{L}$. The logic $\sigma\mathcal{L}$ exists for each $\mathcal{L}$ and it has a simple description: $\Gamma \vdash_{\sigma\mathcal{L}} \varphi$ if and only if the rule $\Gamma \vdash \varphi$ is *admissible*, that is, for each substitution $\sigma$ we have $\emptyset \vdash_{\mathcal{L}} \sigma\varphi$ whenever $\emptyset \vdash_{\mathcal{L}} \sigma\gamma$ for each $\gamma \in \Gamma$.

Antistructural completions involve the same notions, but with respect to antitheorems rather than theorems. Here some clarification is in order: an *antitheorem* of $\mathcal{L}$ is a set of formulas $\Gamma$ such that no valuation into a model of $\mathcal{L}$ designates each $\gamma \in \Gamma$. Equivalently, $\Gamma$ is an antitheorem of $\mathcal{L}$ (symbolically, $\Gamma \vdash_{\mathcal{L}} \emptyset$) if $\sigma\Gamma \vdash_{\mathcal{L}} \mathrm{Fm}_{\mathcal{L}}$ for each substitution $\sigma$, where $\mathrm{Fm}_{\mathcal{L}}$ is the set of all formulas of $\mathcal{L}$. A set of formulas $\Gamma$ is an antitheorem of $\mathcal{L}$ if $\Gamma \vdash_{\mathcal{L}} \mathrm{Fm}$ *provided that $\mathcal{L}$ has an antitheorem* (or provided that $\Gamma$ is finite). It may happen, however, that a logic has no antitheorems, e.g. the positive fragment of classical or intuitionistic logic.

The *antistructural completion* of a logic $\mathcal{L}$ is defined as the largest logic $\alpha\mathcal{L}$ (whenever it exists) which has the same antitheorems as $\mathcal{L}$. Naturally, a logic $\mathcal{L}$ is then *antistructurally complete* if $\alpha\mathcal{L} = \mathcal{L}$. As a first example, consider intuitionistic logic $\mathcal{IL}$. Its antistructural completion may be computed using Glivenko's theorem. We have:

$$\Gamma \vdash_{\mathcal{IL}} \emptyset \;\Leftrightarrow\; \emptyset \vdash_{\mathcal{IL}} {\sim}\bigwedge\Gamma \;\Leftrightarrow\; \emptyset \vdash_{\mathcal{CL}} {\sim}\bigwedge\Gamma \;\Leftrightarrow\; \Gamma \vdash_{\mathcal{CL}} \emptyset$$

for finite $\Gamma$, hence $\mathcal{IL}$ and $\mathcal{CL}$ have the same antitheorems. Therefore classical logic $\mathcal{CL}$ is the antistructural completion of $\mathcal{IL}$ by virtue of being its largest non-trivial extension. Our aim will be to generalize this Glivenko-like connection between $\mathcal{IL}$ and $\mathcal{CL}$ to a wider setting.

For this purpose, the following notion is the natural counterpart of admissibility. A rule $\Gamma \vdash \varphi$ will be called *antiadmissible* in $\mathcal{L}$ if for each substitution $\sigma$ and each $\Delta$ we have:

$$\sigma\Gamma, \Delta \vdash_{\mathcal{L}} \emptyset \text{ whenever } \sigma\varphi, \Delta \vdash_{\mathcal{L}} \emptyset$$

**Lemma.** *The antiadmissible rules of each logic form a reflexive monotone structural relation which is closed under finitary cuts (but not necessarily under arbitrary cuts).*

However, unlike the admissible rules, the antiadmissible rules in general need not define a logic and the antistructural completion of a logic need not exist.

**Example.** *Consider the standard Gödel chain $[0,1]_G$ expanded by a constant $c_q$ for each rational $q \in \mathbb{Q} \cap [0,1]$. The logic defined semantically by all the* principal *filters on this chain does not have an antistructural completion.*

The existence of antistructural completions is therefore a rather more delicate matter than in the case of structural completions. Our main result now provides a widely applicable sufficient condition for the existence of $\alpha\mathcal{L}$ and several equivalent descriptions of this logic.

It involves a technical property which we call the *maximal consistency property (MCP)* which states that each consistent theory, i.e. a theory $\Gamma$ such that $\Gamma \nvdash \emptyset$, may be extended to a maximal consistent theory. In particular, each finitary logic enjoys this property.

**Theorem.** *Let $\mathcal{L}$ be a logic with a finite antitheorem which enjoys the MCP. (For example, let $\mathcal{L}$ be a finitary logic with an antitheorem.) Then $\alpha\mathcal{L}$ exists and the following are equivalent:*

*(i)* $\Gamma \vdash_{\alpha\mathcal{L}} \varphi$.

*(ii)* $\Gamma \vdash \varphi$ *is antiadmissible in* $\mathcal{L}$.

*(iii)* $\Gamma \vdash \varphi$ *is valid in all* $\mathcal{L}$*-models* $\langle \boldsymbol{Fm}, \Gamma \rangle$ *where* $\Gamma$ *is a maximal consistent theory.*

*If* $\mathcal{L}$ *is moreover protoalgebraic, then these are equivalent to:*

*(iv)* $\sigma\varphi, \Delta \vdash_{\mathcal{L}} \emptyset$ *implies* $\sigma\Gamma, \Delta \vdash_{\mathcal{L}} \emptyset$ *for each* $\Delta$ *and each invertible substitution* $\sigma$.

*(v)* $\Gamma \vdash \varphi$ *is valid in all (reduced)* $\kappa$*-generated* $\mathcal{L}$*-simple matrices for* $\kappa = |\operatorname{Var}_{\mathcal{L}}|$.

*If* $\mathcal{L}$ *enjoys the local deduction theorem (LDDT) and finitarity, then these are equivalent to:*

*(vi)* $\varphi, \Delta \vdash_{\mathcal{L}} \emptyset$ *implies* $\Gamma, \Delta \vdash_{\mathcal{L}} \emptyset$ *for each* $\Delta$.

*(vii)* $\Gamma \vdash \varphi$ *is valid in all (reduced)* $\mathcal{L}$*-simple matrices.*

**Proposition.** *A finitary logic with an antitheorem* $\mathcal{L}$ *which enjoys the LDDT is antistructurally complete if and only if* $\operatorname{Mod}\mathcal{L}$ *is semisimple (each subdirectly irreducible* $\mathcal{L}$*-model is* $\mathcal{L}$*-simple).*

Item (iv) above may in fact be replaced by item (vi) whenever $\Gamma$ is finite, or more generally whenever there are at least $\kappa$ variables which do not occur in $\Gamma$ for $\kappa = |\operatorname{Var}_{\mathcal{L}}|$.

Let us now provide some examples of antistructural completions of known logics to illustrate this notion. The following two claims are essentially reformulations of the results of [1] and [3].

**Example.** *The antistructural completion of Hájek's Basic Fuzzy Logic is the (infinitary) Łukasiewicz logic. Each axiomatic extension of the Full Lambek calculus with exchange and weakening which validates the axiom* $p \vee \neg(p^n)$ *for some* $n \in \omega$ *is antistructurally complete.*

Our main result has some use even outside the realm of protoalgebraic logics.

**Example.** *The antistructural completion of the four-valued Belnap–Dunn logic* $\mathcal{B}$ *is Priest's three-valued Logic of Paradox. The antistructural completion of the extension of* $\mathcal{B}$ *by the rule* $p, \neg p \vdash q$ *is the Exactly True Logic, i.e. the extension of* $\mathcal{B}$ *by the rule* $p, \neg p \vee q \vdash q$.

Observe also that the same notions can be considered for algebras rather than logics. For example, the variety of De Morgan algebras is antistructurally complete, while the antistructural completion of the variety of De Morgan lattices is the variety of Kleene lattices.

# References

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