Canonical extensions of archimedean vector lattices with strong order unit

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Canonical extensions of Boolean algebras with operators were introduced in the seminal paper of Jónsson and Tarski [7]. They were generalized to distributive lattices with operators [4, 5], lattices with operators [2], and further to posets [6, 3].

Stone duality provides motivation for the definition of the canonical extension. For example, the canonical extension $B$ of a Boolean algebra $A$ is isomorphic to the powerset of the Stone space $X$ of $A$, and the embedding $e : A \to B$ is realized as the inclusion of the Boolean algebra $\text{Clop}(X)$ of clopen subsets of $X$ into the powerset $\wp(X)$. The inclusion $\text{Clop}(X) \hookrightarrow \wp(X)$ is dense and compact, and these are the defining properties of the canonical extension:

**Definition 1.** The canonical extension of a Boolean algebra $A$ is a pair $A^\sigma = (B, e)$, where $B$ is a complete Boolean algebra and $e : A \to B$ is a Boolean monomorphism satisfying:

1. (Density) Each $x \in B$ is a join of meets and a meet of joins of elements of $e[A]$.

2. (Compactness) For $S, T \subseteq A$, from $\bigwedge e[S] \leq \bigvee e[T]$ it follows that $\bigwedge e[S'] \leq \bigvee e[T']$ for some finite $S' \subseteq S$ and $T' \subseteq T$.

A similar situation arises for archimedean vector lattices with strong order unit. Let $A$ be an archimedean vector lattice with strong order unit. By Yosida representation [8], $A$ is represented as a uniformly dense vector sublattice of the vector lattice $C(Y)$ of all continuous real-valued functions on the Yosida space $Y$ of $A$. Moreover, if $A$ is uniformly complete, then $A$ is isomorphic to $C(Y)$. Since $Y$ is compact, every continuous real-valued function on $Y$ is bounded. Therefore, $C(Y)$ is a vector sublattice of the vector lattice $B(Y)$ of all bounded real-valued functions on $Y$.

The inclusion $C(Y) \hookrightarrow B(Y)$ has many similarities with the inclusion $\text{Clop}(X) \hookrightarrow \wp(X)$. In particular, $C(Y)$ is dense in $B(Y)$. However, it is never compact in the sense of Definition 1. Indeed, if $Y$ is a singleton, then both $C(Y)$ and $B(Y)$ are isomorphic to $\mathbb{R}$. Now, if $S = \{\beta \in \mathbb{R} : 1/2 < \beta \leq 1\}$ and $T = \{\alpha \in \mathbb{R} : 0 \leq \alpha < 1/2\}$, then $\bigwedge S \leq \bigvee T$ as both are $1/2$, but there are not finite subsets $S' \subseteq S$ and $T' \subseteq T$ with $\bigwedge S' \leq \bigvee T'$.

Our goal is to tweak the definition of compactness appropriately, so that coupled with density, it captures algebraically the behavior of the inclusion $C(Y) \hookrightarrow B(Y)$.

Let $A$ be an archimedean vector lattice and let $u \in A$ be the strong order unit of $A$. We identify $\mathbb{R}$ with a subalgebra of $A$ by identifying $\alpha \in \mathbb{R}$ with $\alpha u \in A$.

**Definition 2.** The canonical extension of an archimedean vector lattice with strong order unit $A$ is a pair $A^\sigma = (B, e)$, where $B$ is a Dedekind complete (archimedean) vector lattice with strong order unit and $e : A \to B$ is a unital vector lattice monomorphism satisfying:

1. (Density) Each $x \in B$ is a join of meets and a meet of joins of elements of $e[A]$.

2. (Compactness) For $S, T \subseteq A$ and $0 < \varepsilon \in \mathbb{R}$, from $\bigwedge e[S] + \varepsilon \leq \bigvee e[T]$ it follows that $\bigwedge e[S'] \leq \bigvee e[T']$ for some finite $S' \subseteq S$ and $T' \subseteq T$. 

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Theorem 3. Let $X$ be a completely regular space, and let $C^*(X)$ be the vector lattice of bounded continuous real-valued functions on $X$. Then $B(X)$ is the canonical extension of $C^*(X)$ if and only if $X$ is compact.

Regardless of whether $X$ is compact, the vector lattice $C^*(X)$ is dense in $B(X)$ in the sense of Definition 2. Thus, the theorem shows that the compactness axiom of Definition 2 when applied to $C^*(X)$ and $B(X)$ gives an algebraic formulation of topological compactness.

Theorem 4. Let $A$ be an archimedean vector lattice with strong order unit, $Y$ the Yosida space of $A$, and $e : A \to C(Y)$ the Yosida embedding. Then the pair $(B(Y), e)$ is up to isomorphism the canonical extension of $A$. Thus, canonical extensions of archimedean vector lattices with strong order unit always exist and are unique up to isomorphism.

In fact, the correspondence $A \mapsto A^*$ is functorial. This functoriality of canonical extensions contrasts with the lack of it for Dedekind completions [1].

It is well known that a Boolean algebra can be realized as the canonical extension of some other Boolean algebra if and only if it is complete and atomic. We give a similar characterization in our setting. Suppose that $B$ is an archimedean vector lattice with strong order unit. If $B$ is Dedekind complete, then it has a unique multiplication which makes it a lattice-ordered ring (see, e.g., [1, Sec. 8]). Viewing $B$ as a ring, since $B$ is Dedekind complete, the idempotents $\text{Id}(B)$ of $B$ form a complete Boolean algebra. Then $B$ is a canonical extension of some vector lattice $A$ with strong order unit if and only if $\text{Id}(B)$ is atomic. We also give a purely ring-theoretic characterization of $B$ as a Baer ring with essential socle.

References


