Canonical extensions of archimedean vector lattices with strong order unit

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Canonical extensions of Boolean algebras with operators were introduced in the seminal paper of Jónsson and Tarski [7]. They were generalized to distributive lattices with operators [4, 5], lattices with operators [2], and further to posets [6, 3].

Stone duality provides motivation for the definition of the canonical extension. For example, the canonical extension B of a Boolean algebra A is isomorphic to the powerset of the Stone space X of A, and the embedding $e : A \to B$ is realized as the inclusion of the Boolean algebra $\mathsf{Clop}(X)$ of clopen subsets of X into the powerset $\wp(X)$. The inclusion $\mathsf{Clop}(X) \hookrightarrow \wp(X)$ is dense and compact, and these are the defining properties of the canonical extension:

Definition 1. The *canonical extension* of a Boolean algebra A is a pair $A^\sigma = (B, e)$, where B is a complete Boolean algebra and $e : A \to B$ is a Boolean monomorphism satisfying:

1. (Density) Each $x \in B$ is a join of meets and a meet of joins of elements of $e[A]$.
2. (Compactness) For $S, T \subseteq A$, from $\bigwedge e[S] \leq \bigvee e[T]$ it follows that $\bigwedge e[S'] \leq \bigvee e[T']$ for some finite $S' \subseteq S$ and $T' \subseteq T$.

A similar situation arises for archimedean vector lattices with strong order unit. Let A be an archimedean vector lattice with strong order unit. By Yosida representation [8], A is represented as a uniformly dense vector sublattice of the vector lattice $C(Y)$ of all continuous real-valued functions on the Yosida space Y of A. Moreover, if A is uniformly complete, then A is isomorphic to $C(Y)$. Since Y is compact, every continuous real-valued function on Y is bounded. Therefore, $C(Y)$ is a vector sublattice of the vector lattice $B(Y)$ of all bounded real-valued functions on Y.

The inclusion $C(Y) \hookrightarrow B(Y)$ has many similarities with the inclusion $\mathsf{Clop}(X) \hookrightarrow \wp(X)$. In particular, $C(Y)$ is dense in $B(Y)$. However, it is never compact in the sense of Definition 1. Indeed, if Y is a singleton, then both $C(Y)$ and $B(Y)$ are isomorphic to $\mathbb{R}$. Now, if $S = \{\beta \in \mathbb{R} : 1/2 < \beta \leq 1\}$ and $T = \{\alpha \in \mathbb{R} : 0 \leq \alpha < 1/2\}$, then $\bigwedge S \leq \bigvee T$ as both are $1/2$, but there are not finite subsets $S' \subseteq S$ and $T' \subseteq T$ with $\bigwedge S' \leq \bigvee T'$.

Our goal is to tweak the definition of compactness appropriately, so that coupled with density, it captures algebraically the behavior of the inclusion $C(Y) \hookrightarrow B(Y)$.

Let A be an archimedean vector lattice and let $u \in A$ be the strong order unit of A. We identify $\mathbb{R}$ with a subalgebra of A by identifying $\alpha \in \mathbb{R}$ with $\alpha u \in A$.

Definition 2. The *canonical extension* of an archimedean vector lattice with strong order unit A is a pair $A^\sigma = (B, e)$, where B is a Dedekind complete (archimedean) vector lattice with strong order unit and $e : A \to B$ is a unital vector lattice monomorphism satisfying:

1. (Density) Each $x \in B$ is a join of meets and a meet of joins of elements of $e[A]$.
2. (Compactness) For $S, T \subseteq A$ and $0 < \varepsilon \in \mathbb{R}$, from $\bigwedge e[S] + \varepsilon \leq \bigvee e[T]$ it follows that $\bigwedge e[S'] \leq \bigvee e[T']$ for some finite $S' \subseteq S$ and $T' \subseteq T$.

Theorem 3. *Let X be a completely regular space, and let $C^*(X)$ be the vector lattice of bounded continuous real-valued functions on X. Then $B(X)$ is the canonical extension of $C^*(X)$ if and only if X is compact.*

Regardless of whether X is compact, the vector lattice $C^*(X)$ is dense in $B(X)$ in the sense of Definition 2. Thus, the theorem shows that the compactness axiom of Definition 2 when applied to $C^*(X)$ and $B(X)$ gives an algebraic formulation of topological compactness.

Theorem 4. *Let A be an archimedean vector lattice with strong order unit, Y the Yosida space of A, and $e : A \to C(Y)$ the Yosida embedding. Then the pair $(B(Y), e)$ is up to isomorphism the canonical extension of A. Thus, canonical extensions of archimedean vector lattices with strong order unit always exist and are unique up to isomorphism.*

In fact, the correspondence $A \mapsto A^\sigma$ is functorial. This functoriality of canonical extensions contrasts with the lack of it for Dedekind completions [1].

It is well known that a Boolean algebra can be realized as the canonical extension of some other Boolean algebra if and only if it is complete and atomic. We give a similar characterization in our setting. Suppose that B is an archimedean vector lattice with strong order unit. If B is Dedekind complete, then it has a unique multiplication which makes it a lattice-ordered ring (see, e.g., [1, Sec. 8]). Viewing B as a ring, since B is Dedekind complete, the idempotents $\mathrm{Id}(B)$ of B form a complete Boolean algebra. Then B is a canonical extension of some vector lattice A with strong order unit if and only $\mathrm{Id}(B)$ is atomic. We also give a purely ring-theoretic characterization of B as a Baer ring with essential socle.

References

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