Sublocales and a Boolean extension of a frame *

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This talk is about sublocales, the natural subobjects in the category of locales (which one may think about as generalized topological spaces), that is, in the dual category of the category of frames ([3]).

Sublocales of a frame L are well defined subsets of L, and constitute, in the natural inclusion order, a coframe $\mathsf{S}(L)$. Hence sublocale lattices are more complicated than their topological counterparts (complete and atomic Boolean algebras). One of the main differences is that only complemented sublocales (and most sublocales are not complemented) distribute over all joins of sublocales. But, as J. Isbell emphasized, a locale has enough complemented sublocales to compensate for this shortcoming: one has open and closed sublocales (precisely corresponding to classical open and closed subspaces), complementing each other.

A separation axiom called subfitness (making sense for classical spaces as well, slightly weaker than T_1) is characterized by the property that *every open sublocale is a join of closed ones*, and another, stronger, called fitness (akin to regularity) is characterized by the fact that *every closed sublocale is an intersection of open ones.* These properties sound dual to each other, but is not quite so: in fact *in a fit frame every sublocale whatsoever is an intersection of open ones* which has no counterpart in the subfit case. Now what does the property that *every sublocale whatsoever is a join of closed ones* mean? In [1] it was shown that it characterizes the so called scattered frames (quite analogous to scattered topological spaces), formally the L with Boolean $\mathsf{S}(L)$. The main goal of our talk will be to discuss the system $\mathsf{S}_\mathfrak{c}(L)$ of all the sublocales of a general L that are joins of closed ones.

We will start the talk by presenting the basics about $\mathsf{S}_\mathfrak{c}(L)$ ([5]). First, $\mathsf{S}_\mathfrak{c}(L)$ is always a frame. Since it is a join-sublattice, is it not also a coframe, or even a subcolocale of $\mathsf{S}(L)$? We give a complete answer for subfit frames L. There, indeed, $\mathsf{S}_\mathfrak{c}(L)$ is a subcolocale (and in fact this is another characterization of subfitness). Moreover, it is a Boolean algebra and in fact precisely the Booleanization of $\mathsf{S}(L)$. Further, we have here a Boolean extension $L \to \mathsf{S}_\mathfrak{c}(L)$ by *open* sublocales; this is compared with the well known frame extension $L \to \mathsf{S}(L)^{\mathrm{op}}$ by *closed* sublocales (the embedding into the frame of congruences), and the relation is analysed.

Subspaces of a space can be viewed as sublocales (more precisely, sublocales of the associated frame of open sets $\Omega(X)$). But in general there are more sublocales than subspaces (a space has typically generalized subspaces that are not classical induced ones). In case of a T_1-space X, it turns out that the classical ones constitute precisely the $\mathsf{S}_\mathfrak{c}(\Omega(X))$, and hence the Booleanization of $\mathsf{S}(\Omega(X))$.

Point-free modeling of real-valued functions on a frame L that are not necessarily continuous has been so far based on the extension of L to its frame of sublocales $\mathsf{S}(L)^{\mathrm{op}}$, mimicking the replacement of a topological space by its discretization ([2]). If time permits, we will explain why the smaller $\mathsf{S}_\mathfrak{c}(L)$ can replace $\mathsf{S}(L)^{\mathrm{op}}$ with advantages in the case of a subfit L ([4]).

*A part of the talk is joint work with Anna Tozzi [5].

References

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